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THE DEVELOPMENT OF ANALYSIS OF VARIANCE TECHNIQUES FOR ANGULAR DATA

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THE DEVELOPMENT OF ANALYSIS OF VARIANCE TECHNIQUES FOR ANGULAR DATA

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ABSTRACT

In many areas of research, such as within medical statistics, biology and geostatistics, problems arise requiring the analysis of angular (or directional) data. Many possess experimental design problems and require analysis of variance techniques for suitable analysis of the angular data. These techniques have been developed for very limited cases and the sensitivity of such techniques to the violation of assumptions made, and their possible extension to larger experimental models, has yet to be investigated.

The general aim of this project is therefore to develop suitable experimental design models and analysis of variance type techniques for the analysis of directional data.

Initially a generalised linear modelling approach is used to derive parameter estimates for one-way classification designs leading to maximum likelihood methods. This approach however, when applied to larger experimental designs is shown to be intractable due to optimization problems.

The limited analysis of variance techniques presently available for angular data are reviewed and extended to take account of the possible addition of further factors within an experimental design. These are shown to breakdown under varying conditions and question basic underlying assumptions regarding the components within the original approach.

A new analysis of variance approach is developed which possesses many desirable properties held in standard 'linear' statistical analysis of variance.

Finally several data sets are analysed to support the validity of the new techniques.

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CHAPTER 1

INTRODUCTION

1.1 Directional Data

In many scientific fields the experimenter is interested primarily in the direction of a measured variable. These observations will be bearings from some central point, or origin, ending on a sphere or circumference of a circle, and may be regarded as vectors. The radius can be represented as a unit vector, while the length or magnitude of the vector is not important. Directions may be thought of in any number of dimensions but in practice they are invariably collected in two or three dimensional space. The tests and new results presented in this thesis are solely concerned with directions in two-dimensions. Directions are measured by angles ranging from 0° to 360°, or, equivalently, from 0 to 2π radians. Circular or directional data is the name given to data which arise when the observations are angles.

There are many examples of circular data originating from various disciplines. For example, geologists study the orientation of fractures in deformed rocks to interpret structural changes, and the orientation of cross-bedding or particles in undisturbed sediments to the direction of depositing currents of wind and water. (Pincus (1953), Curray (1956), Sengupta and Rao (1966), and Sanderson (1976)). The classic example of directional data is from the study of bird orientation in homing or migration which involves observing the birds vanishing angles from their release point. Zoologists use such data to investigate consistency of bird migration under certain conditions. Many examples from this field of study are well cited and illustrated by Batschelet (1965, 1981).

Directional data is not confined to observations directly measured in degrees or radians, but may also occur in the area of biological rhythms. A period of 24 hours corresponds to a full turn of 360 degrees. Similarly, a month, a year or any other period of a cyclic event may be represented by a rotation of 360 degrees. The number of deaths due to a disease or the number of onsets of a disease in each month over years fall in this category and can be treated as directional or circular data. Other examples of this type can be seen in Gumbel (1954).

It is tempting to use the conventional measures of location and spread used in linear analysis to analyse directional data. For example, suppose our data are the four values 5, 14, 351 and 10, a simple arithmetic mean would give a value of 95. For linear analysis this is understandable as the value of 351 has a large influence and draws the mean away from the other data points. If these values are now regarded as angles the spread of the whole sample is reduced, since in angular terms the sample value of 351° is now situated close to the other data points. Similarly the point of central location will have changed considerably and can now be seen to be around zero degrees. Then the simple arithmetic mean would not, in general, give a meaningful mean direction of the sample, similarly, the standard deviation would not give a good measure of dispersion. If, however, the zero direction was taken at a different position on the circle such as the y–axis in place of the x–axis then the linear measure may give a sensible result. For example, if the above sample values were rotated by 90°, to become 95°, 104°, 81° and 100°, the arithmetic mean would give a sensible result of 95°. It is therefore not possible to define an arithmetic mean or standard deviation in such a way that it is invariant under a rotation of the circle. This heavy dependence on the zero direction shows the inappropriate use of basic linear methods for circular statistics. Simple examples of such problems are given by Batschelet (1965, 1981), Mardia (1972) and Watson (1983). Distribution functions, characteristic functions and moments all suffer from the same draw–back and must in some way take account of the natural periodicity of the circle.

1.2 A General Thesis Review

Having introduced the study of directional data, this section gives a brief review of the work discussed within each of the following chapters, whilst indicating the structure and progression of the thesis as a whole. The following two sections within this chapter give further background to the subject of directional data. The first discusses different types of probability distributions that may exist on the circle, whilst the second states the elementary statistics of angular data required for further use in later chapters.

In Chapter 2 the von Mises distribution is discussed from estimation and distributional view points. The maximum likelihood estimates of the von Mises parameters are seen to be asymptotically independent so that construction of simple large-sample tests, for differing hypotheses regarding the parameters, may be carried out. Interest is focused on the distributional form of the resultant length on the random (Uniform) distribution, $k = 0$, and the von Mises distribution. The results from the special case of the random distribution are required since terms in its solution arise again in the general distribution theory. A summary of exact and approximate moments of the resultant length, R, are given in preparation for the derivation of further circular statistical tests.

Chapter 3 is a review of the maximum likelihood results and tests for the von Mises distribution. Extensive investigation of the many approximations for the von Mises concentration parameter, k, for both small and large k, has been undertaken. Seven approximations for the maximum likelihood estimate of small k, $\hat{k}$, which have been cited by various authors are reviewed. Their corresponding values, residuals and relative residuals are plotted to enable comparison and evaluation of their accuracy. Similar work is carried out for nine approximations to large k, $\hat{k}$. A summary of the 'best' and 'best simple' approximations is given in Section 3.5 against varying ranges of the concentration parameter.

Chapter 4 gives an historical review of the development of analysis of variance techniques. The work covers the exact and approximate tests for differing mean directions derived by Watson (1956), Watson and Williams (1956) and Stephens (1962a, b and c), to multi-sample tests for the equality of concentration parameters. The homogeneity tests for varying ranges of concentration parameter are cited for later use with new design tests.

Chapter 5 discusses the use of the generalised linear modelling approach for circular statistics to derive parameter estimates leading to maximum likelihood methods. For circular statistics it is shown to be desirable to choose the constraint on the angles specifying the factor parameters so that their sines sum to zero. Section 5.3 shows how parameter estimates for the one-way classification design may be found and therefore assist in further understanding of the underlying structure under investigation. The approach, however, when applied to larger experimental designs is seen, at this time, to be intractable since the optimization procedures cannot be solved due to the numerous local maxima found within the constrained equations.

Chapters 6 and 7 examine the possibility of extending the original procedure for the one-way analysis, derived by Watson and Williams (1956), to larger designs, for large k. Chapter 6 shows the construction of the nested or hierarchical design, the randomised complete block and two-way classification design with interaction together with a comparison of their accuracy to the chi-squared and F distributions. Chapter 7, however, shows the possible collapse of these test statistics under particular circumstances. The problems associated with the combining of circular mean directions are shown to be influential in this collapse whilst the cross-product terms are seen to be non-zero and requiring a correction factor to eliminate them.

Chapter 8 develops a new analysis of variance approach by taking account of the resultant lengths together with their corresponding mean directions to eliminate the possible collapse discussed in Chapter 7. The method is still based on maximum likelihood techniques but requires the user to test for equality of concentration parameters prior to testing for any difference between mean directions. The cross-product terms are examined and found to equal their desired combined value of zero. An investigation of the interpretation and representation of interaction on the circle is given in Section 8.4 prior to its calculation via the new approach. For the two-way design the cross-product terms are again shown to equal zero. Further designs are then constructed in the same manner.

Following the development of the procedures in Chapter 8, Chapter 9 examines the statistical theory and distributions behind the new design components and test statistics. The exact theoretical distributions are seen to be intractable, and therefore distribution approximations are used to examine the theory whilst simulation techniques reproduce the distributions of the test statistics for comparison with their assumed expected distributions. The comparisons are carried out for both large and small k and test statistic improvements are made using the component moments. The power of the new tests are also compared with existing tests for the multi-sample case and are seen to compare favourably for both large and small k.

Chapter 10 reproduces the components within the new procedure for the randomised complete block and two-way designs together with their improvement factor derived in Chapter 9. The component statistics and test statistics are compared to their respective exact chi-squared and F distributions. These two designs are used to illustrate the validity of the approach for larger more complex design situations.

Chapter 11 gives several examples where the new approach is applied to real data sets with varying sizes of concentration parameter. The examples vary from the one-way design to the Graeco-Latin square and split plot designs.

Finally Chapter 12 summarises the development of the new analysis of variance techniques. The adequacy of the new procedures, produced for both large and small concentration parameter, are discussed together with their respective components and test statistics.

Appendix A gives a list of notations used throughout the thesis together with the design notations set out in tabular form. Appendix B reviews the techniques used to simulate the von Mises distribution and the required experimental designs. The size and accuracy of the numerical results are also discussed.

1.3 Probability Distributions on the Circle

There is no single distribution on the circle which has all the desirable properties which the Normal distribution possesses on the line. Most of the distributions on the circle have been derived either from transformations of the standard univariate (or bivariate) distributions or as circular analogues of important univariate characteristics. Linear distributions may have a finite range, range to infinity, or may even extend over the whole straight line. Circular distributions, however, are always finite ranging from 0° to 360° (or, equivalently, 0 to 2π), or are fractions within this range.

In general, circular distributions are continuous over the circumference of the circle and may be specified by a probability density function $f(\theta)$, which is a periodic function satisfying

$$\int_0^{2\pi} f(\theta)\, d\theta = 1 \qquad (1.3.1)$$

Although no circular distribution holds all the desirable properties seen in the Normal distribution, the von Mises distribution (originally referred to as the Circular Normal distribution) is the most generally used distribution in statistical inference on the circle. The importance of the von Mises distribution on the circle is often compared to that of the Normal distribution on the line.

The distribution has probability density function

$$f(\theta) = \frac{1}{2\pi I_0(k)} \exp[k \cos(\theta - \mu_0)] \qquad 0 < \theta \leqslant 2\pi \qquad (1.3.2)$$

where $I_0(k)$ is a modified Bessel function, and k is a parameter of concentration of the data about a mean direction μ_0. A complete discussion of the von Mises and its properties can be seen in Chapter 2.

There are two limiting cases of circular distributions. The first case occurs when all the angles are the same i.e. concentrated at one single point $\theta = \mu_0$. The 'point' distribution has little if no practical or theoretical interest here, but has been used for the analysis of Brownian movement and the paths of beta rays.

The second limiting case is the Uniform distribution, where every angle on the unit circle has an equal chance of occurring or no sector is preferred to any other sector. The probability density of θ is constant over the whole circumference and is defined by

$$f(\theta) = \frac{1}{2\pi} \qquad 0 < \theta \leqslant 2\pi \qquad (1.3.3)$$

As there is no concentration of points about any given direction on the circle, then no mean direction exists. Polya (1935) used an analogue of this when he investigated whether the stars are distributed at random over the celestial sphere.

As in the linear case an infinite number of circular distributions exist. Among these, a few, possessing some desirable properties, have received attention. After the von Mises, probably the most important is the wrapped Normal distribution. This distribution is a natural conversion of the Normal distribution and is obtained, as its name suggests, by wrapping the Normal distribution around the circumference of the circle and adding those probabilities that fall into the same sector of the circle. The addition of the overlapping tails leads to a rather complicated density function, though when σ is small the distribution will be approximately Gaussian in $(0,2\pi)$. Like the Circular Uniform distribution the wrapped Normal distribution possess the additive property i.e. the sum of two or more wrapped Normal distributions produces another wrapped Normal distribution with related parameters. The probability density of a random variable θ, with mean angle $\mu_0 = 0^\circ$, from the wrapped Normal distribution is

$$f(\theta) = \frac{1}{\sigma\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \exp\left[\frac{-(\theta + 2\pi n)^2}{2\sigma^2}\right] \qquad 0 < \theta \leqslant 2\pi \qquad (1.3.4)$$

where the mean vector length is

$$\rho = \exp\left[\frac{-\sigma^2}{2}\right]$$

As ρ tends to zero, the wrapped Normal distribution approaches the Uniform distribution, and as ρ tends to one it is concentrated at a single point. The distribution has applications in the study of diffusion processes, and amongst others has been examined by Stephens (1963) and Bingham (1971).

Another distribution which has been wrapped around the circle in a similar manner to the Normal is the Cauchy distribution. The result is again a unimodal and symmetric circular distribution possessing the additive property. With mean angle μ_0 and mean vector length ρ the probability density function for the wrapped Cauchy is

$$f(\theta) = \frac{1}{2\pi} \left\{ \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos(\theta - \mu_0)} \right\} \tag{1.3.5}$$

introduced by Levy (1939) and studied by Wintner (1947).

Other circular distributions of less importance include the cosine distribution, also called the sine wave distribution, with mean vector length ρ and mean angle μ_0, and has the density function

$$f(\theta) = \frac{1}{2\pi} + \frac{\rho}{\pi} \cos(\theta - \mu_0) \tag{1.3.6}$$

and the cardioid distribution with the density function

$$f(\theta) = \frac{1}{2\pi} [1 + 2\rho \cos(\theta - \mu_0)] \tag{1.3.7}$$

introduced by Jeffreys (1948).

All the circular distributions discussed so far have been unimodal distributions, with a single preferred direction of μ_0. There are, however, circular distributions with multimodal directions. (1.3.8) gives the density function of a multimodal von Mises distribution where v denotes the number of modes.

$$f(\theta) = \frac{1}{2\pi I_0(k)} \exp[k \cos v(\theta - \mu_0)] \tag{1.3.8}$$

This was suggested by Breitenberger (1963) and further investigated by Stephens (1965).

Many examples of bimodal data can be found, particularly in scientific fields where orientation is measurable but direction is not. Batschelet (1965, 1981) gives many examples of bimodal data from animal orientation and navigation. A similar situation occurs if we observe the position of undirected straight lines or undirected axes. Gadsden and Kanji (1983) collected this type of data on clay particles following the removal of their electric charge. These particles do not have a 'head' or 'tail' and so the observations lie in the range 0 to π radians.

Other unimodal circular distributions were discussed by Mardia (1972 p.48–61). Batschelet (1981 p.275–90) also reviews skewed, flat–topped and sharply peaked circular distributions.

The most widely used circular distribution, however, is the von Mises distribution and it is on this distribution that this thesis is based. The following chapters investigate and extend the theory and uses of this distribution.

1.4 Statistics of a Circular Distribution

As we noted in Section 1.1 the simple arithmetic mean would not, in general, give a meaningful mean direction of a sample of angles $\theta_1, \theta_2, \ldots, \theta_N$. The mean direction in circular statistics is determined by applying trigonometric functions.

Let P_i be one of the N observed angles θ_i, i = 1, 2,...,N, with origin 0. Let c_i and s_i be the rectangular components of P_i. Then by definition of sine and cosine,

$$c_i = \cos\theta_i \qquad s_i = \sin\theta_i \tag{1.4.1}$$

where

$$\bar{C} = \frac{1}{N}\sum_{i=1}^{N}\cos\theta_i \qquad \bar{S} = \frac{1}{N}\sum_{i=1}^{N}\sin\theta_i \tag{1.4.2}$$

Therefore, if R is the length of the resultant vector with components $C = \sum c_i$ and $S = \sum s_i$, and r is the length of the mean direction, with components $\bar{C}$ and $\bar{S}$, then

$$r = (\bar{C}^2 + \bar{S}^2)^{\frac{1}{2}} \tag{1.4.3}$$

$$R = \Big[\Big(\sum_{i=1}^{N} c_i\Big)^2 + \Big(\sum_{i=1}^{N} s_i\Big)^2\Big]^{\frac{1}{2}} \qquad R = Nr \tag{1.4.4}$$

Applying basic trigonometry to calculate $\overline{\theta}$, the mean direction of the sample

$$\overline{\theta} = \begin{cases} \arctan\left[\dfrac{\overline{S}}{\overline{C}}\right] & \text{if } \overline{C} > 0 \\ 180^\circ + \arctan\left[\dfrac{\overline{S}}{\overline{C}}\right] & \text{if } \overline{C} < 0 \end{cases} \tag{1.4.5}$$

with the exceptional cases of

$$\overline{\theta} = \begin{cases} 90^\circ \text{ if } \overline{C} = 0 \text{ and } \overline{S} > 0 \\ 270^\circ \text{ if } \overline{C} = 0 \text{ and } \overline{S} < 0 \\ \text{undetermined if } \overline{C} = 0 \text{ and } \overline{S} = 0 \end{cases} \tag{1.4.6}$$

Figure 1.4.1 illustrates these circular measures:

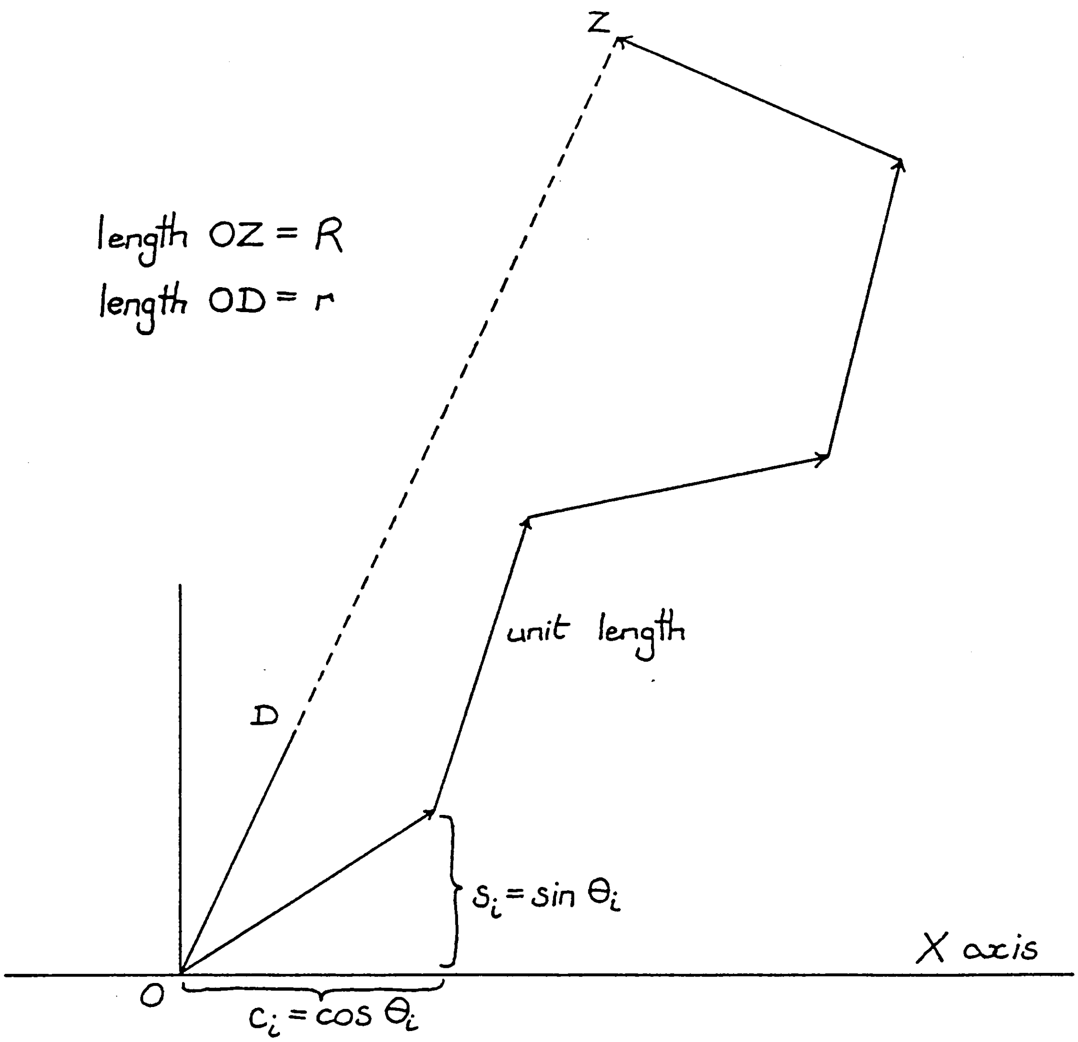


Figure 1.4.1 The Circular Statistics

It is clear from Figure 1.4.1 that the length of the resultant cannot exceed N and similarly, from (1.4.4), that r cannot exceed 1.

Hence,

$$0 \leqslant R \leqslant N \tag{1.4.7}$$

and

$$0 \leqslant r \leqslant 1 \tag{1.4.8}$$

In the extreme case when all sample points fall onto the same point, the length of the mean vector, r, equals 1. When points are close together, concentrated over a small arc, the centre of mass is still very close to the circumference of the unit circle, and r is close to 1. Less concentration leads to smaller values of r. At the lower end r = 0, with no concentration around a single direction. Hence, in unimodal samples, the mean vector length, r, serves as a measure of concentration.

From the above results we may now see that $\bar{\theta}$ has some desirable properties as a measure of location. One such property is that the mean vector does not depend on the zero direction of the sample. If a rotation of ψ is applied to each angle, then the sample values θ_i turn into $\theta_i' = \theta_i - \psi$. Similarly the new mean angle is $\bar{\theta}' = \bar{\theta} - \psi$, but the mean vector length, r, remains invariant.

Examining the sine of the difference between the mean angular direction and the sample angles, it is easily shown that

$$\sum_{i=1}^{N} \sin(\theta_i - \bar{\theta}) = 0 \tag{1.4.9}$$

which is analogous to

$$\sum_{i=1}^{N} (x_i - \bar{x}) = 0 \tag{1.4.10}$$

in linear statistical analysis.

Similarly, for the cosine of the difference

$$\sum_{i=1}^{N} \cos(\theta_i - \bar{\theta}) = R \tag{1.4.11}$$

and therefore

$$\frac{1}{N} \sum_{i=1}^{N} 2[1 - \cos(\theta_i - \bar{\theta})] = 2(1 - r) \tag{1.4.12}$$

For small deviations, $\theta_i - \bar{\theta}$

$$2[1 - \cos(\theta_i - \bar{\theta})] \approx (\theta_i - \bar{\theta})^2 \tag{1.4.13}$$

hence,

$$\frac{1}{N} \sum_{i=1}^{N} (\theta_i - \bar{\theta})^2 \approx 2(1 - r) \tag{1.4.14}$$

which is analogous to

$$\frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2 = s^2 \tag{1.4.15}$$

in linear statistics.

Equation (1.4.12) may be defined as the angular variance and, from (1.4.14), is asymptotically equivalent to the variance in linear statistics.

Taking the square root of (1.4.12) gives a measure of dispersion, equivalent to the standard deviation

$$s = [2(1 - r)]^{\frac{1}{2}} \tag{1.4.16}$$

called the mean angular deviation.

The basic results of this section were adapted from Batschelet (1981) where further discussion of the properties are given. The analogies have been reiterated here so further use may be made of them in later chapters.

CHAPTER 2

THE VON MISES DISTRIBUTION

2.1 Derivation

Gauss showed that the Normal distribution can be derived by the method of maximum likelihood with the single assumption that the mean is the most probable value. Von Mises (1918) applied this to a circular variate, and for this reason Gumbel, Greenwood and Durand (1953) referred to the distribution as the Circular Normal Distribution. Von Mises procedure was for a distribution $f(\theta_i - \mu_0)$, such that the direction μ_0 upon N observations $\theta_1, \theta_2, \ldots\ldots, \theta_N$ is a maximum given by the constraints

$$\sum_{i=1}^{N} \sin(\theta_i - \mu_0) = 0 \tag{2.1.1}$$

and

$$\sum_{i=1}^{N} \frac{f'(\theta_i - \mu_0)}{f(\theta_i - \mu_0)} = 0 \tag{2.1.2}$$

where $f(\theta)$ is the required distribution and $f'(\theta)$ is the first derivative of $f(\theta)$ with respect to μ_0. Since the equations (2.1.1) and (2.1.2) are identical for each θ_i, therefore

$$\frac{f'(\theta - \mu_0)}{f(\theta - \mu_0)} = \sin(\theta - \mu_0) \tag{2.1.3}$$

The equation has the solution

$$f(\theta - \mu_0) = U \exp[k \cos(\theta - \mu_0)] \tag{2.1.4}$$

where the two variables U and k are linked by the condition (1.3.1).

Hence

$$U = \frac{1}{\int_0^{2\pi} \exp[k \cos(\theta - \mu_0)] \, d(\theta - \mu_0)}$$

$$= \frac{1}{2\pi I_0(k)} \tag{2.1.5}$$

where $I_0(k)$ is the modified Bessel function of the first kind and order zero. A proof of (2.1.5) can be seen in Mardia (1972, p.58).

The von Mises distribution then, denoted as $M(\mu_0,k)$, is given by

$$f(\theta) = \frac{1}{2\pi I_0(k)} \exp[k \cos(\theta - \mu_0)] \tag{2.1.6}$$

2.2 Properties of the von Mises Distribution and its Parameters

The von Mises distribution is unimodal and symmetric with its mode at μ_0 and anti-mode at $\mu_0 + \pi$. For $k=0$, the von Mises degenerates into the Uniform distribution, and for large k the distribution concentrates around the mean direction. Therefore, k is called the parameter of concentration. The concentration parameter k is analogous to the inverse of the variance parameter σ^2 of the Normal distribution in its effect on the shape of the distribution. For sufficiently large k we may approximate the von Mises by the Normal distribution. Using an approximation quoted by Bickley (1957), for large k

$$I_0(k) \approx \left[\frac{1}{(2\pi k)^{\frac{1}{2}}}\right]\left[\frac{\exp(k)}{(c(k))}\right]$$

where

$$\frac{1}{c(k)} = 1 + \sum_{r=1}^{\infty} \left\{\frac{(2r-1)!}{2^{r-1}(r-1)!}\right\}^2 \frac{1}{r!\,(2k)^r} \tag{2.2.1}$$

Extending the limits of this approximation would be reasonable since, when k is large, the additional area is negligible. Replacing this into (2.2.2), a von Mises with mean at zero;

$$f(\theta) = \frac{1}{2\pi I_0(k)} \exp(k \cos \theta) \qquad (-\pi \leqslant \theta < \pi) \qquad (2.2.2)$$

gives

$$f(\theta) \approx c(k)\left[\frac{k}{2\pi}\right]^{\frac{1}{2}} \exp\{k((\cos \theta) - 1)\} \qquad (-\infty \leqslant \theta \leqslant \infty) \qquad (2.2.3)$$

Gumbel, Greenwood and Durand (1953) simplified (2.2.3) to obtain

$$M(0,k) \approx N(0,k^{-\frac{1}{2}})$$

alternatively

$$\theta\sqrt{k} \approx N(0,1) \qquad (2.2.4)$$

Upton (1974) considered more accurate approximations by taking further terms in the power series of $((\cos \theta) - 1)$, and produced two new approximations

$$\sqrt{k}\left[1 - \frac{1}{8k}\right]\theta \approx N(0,1) \qquad (2.2.5)$$

and more accurately,

$$\sqrt{k}\left\{\left[1 - \frac{1}{8k}\right]\theta - \frac{1}{24}\left[1 + \frac{1}{4k}\right]\theta^3\right\} \approx N(0,1) \qquad (2.2.6)$$

A fourth approximation was considered by Upton (1974) given by Mardia (1972, p.64), without proof;

$$\left[k - \frac{1}{2}\right]^{\frac{1}{2}}\theta \approx N(0,1) \qquad (2.2.7)$$

Upton tested the power of all the approximations, finding that all four consistently overestimated the upper tail probability. Approximation (2.2.6) was found to be the best, and this was later confirmed from further work by Hill (1978). Stephens

(1962c) gave another approach to the equality of the two distributions using the moment generating function.

Estimates of the parameters k and μ_0 may be obtained by means of the maximum likelihood method.

A sample of N angles $\theta_1, \theta_2, \ldots \theta_N$ are collected from a von Mises distribution with unknown population parameters k and μ_0 which we wish to estimate. The probability density for these angles is

$$c^N \exp[k \cos(\theta_1 - \mu_0)] \exp[k \cos(\theta_2 - \mu_0)] \ldots\ldots$$

$$= c^N \exp k[\cos(\theta_1 - \mu_0) + \cos(\theta_2 - \mu_0) + \ldots\ldots\ldots] \qquad (2.2.8)$$

where

$$c = \frac{1}{2\pi I_0(k)}$$

The log likelihood function is

$$\log L(\mu_0, k) = -N\log I_0(k) + k\sum_{i=1}^{N} \cos(\theta_i - \mu_0) + \text{const} \qquad (2.2.9)$$

For the maximum likelihood estimate of the parameter μ_0

$$\frac{d\log L}{d\mu_0} = k[\sin(\theta_1 - \mu_0) + \sin(\theta_2 - \mu_0) + \ldots\ldots\ldots]$$

$$\frac{d\log L}{d\mu_0} = k\sum_{i=1}^{N} \sin(\theta_i - \mu_0) \qquad (2.2.10)$$

and vanishes for the particular value $(\hat{\mu}_0)$ with

$$\sum_{i=1}^{N} \sin(\theta_i - \hat{\mu}_0) = 0$$

From (1.4.9) we know that this equation is satisfied for the sample mean angle $\bar{\theta}$ which we calculated in (1.4.5). Hence, the maximum likelihood estimate of the parameter μ_0 of a von Mises distribution is the sample mean angle $\bar{\theta}$. Bingham and Mardia (1975) showed that there exists only one circular distribution for which the sample mean angle is the maximum likelihood estimate of the population mean angle, namely the von Mises distribution.

For the maximum likelihood estimate of the parameter k

$$\frac{d\log L}{d(k)} = -N\frac{I_1(k)}{I_0(k)} + \sum_{i=1}^{N}\cos(\theta_i - \mu_0) \tag{2.2.11}$$

Therefore dlog L/d(k) is zero if

$$\frac{I_1(k)}{I_0(k)} = \frac{1}{N}\sum_{i=1}^{N}\cos(\theta_i - \mu_0) \tag{2.2.12}$$

The right hand side is the mean vector length of the sample, r, as indicated by equation (1.4.11). Hence, the maximum likelihood estimate of k is the solution of

$$A(\hat{k}) = \frac{I_1(\hat{k})}{I_0(\hat{k})} = \frac{R}{N} = r \tag{2.2.13}$$

If the mean direction is known to be μ_0, then the maximum likelihood estimate of k, $\hat{k}$, is no longer given by equation (2.2.13), but instead by equation (2.2.14)

$$\frac{I_1(\hat{k})}{I_0(\hat{k})} = \frac{X}{N} \tag{2.2.14}$$

where X is the component of R on $\bar{\theta}$, when μ_0 is known.

The solution of (2.2.14) is obtained numerically. Tables are not provided here since adequate tables have been produced by Upton (1970, Appendix G), Mardia (1972,

Appendix 2.2) and Batschelet (1981, Table B). For the extreme cases, there are approximate solutions to (2.2.13) which will be discussed in Chapter 3.

Figure 2.2.1, taken from Upton (1970), illustrates the measures R and X with their relationship to C and S from (1.4.2). δ is the angle between R and X.

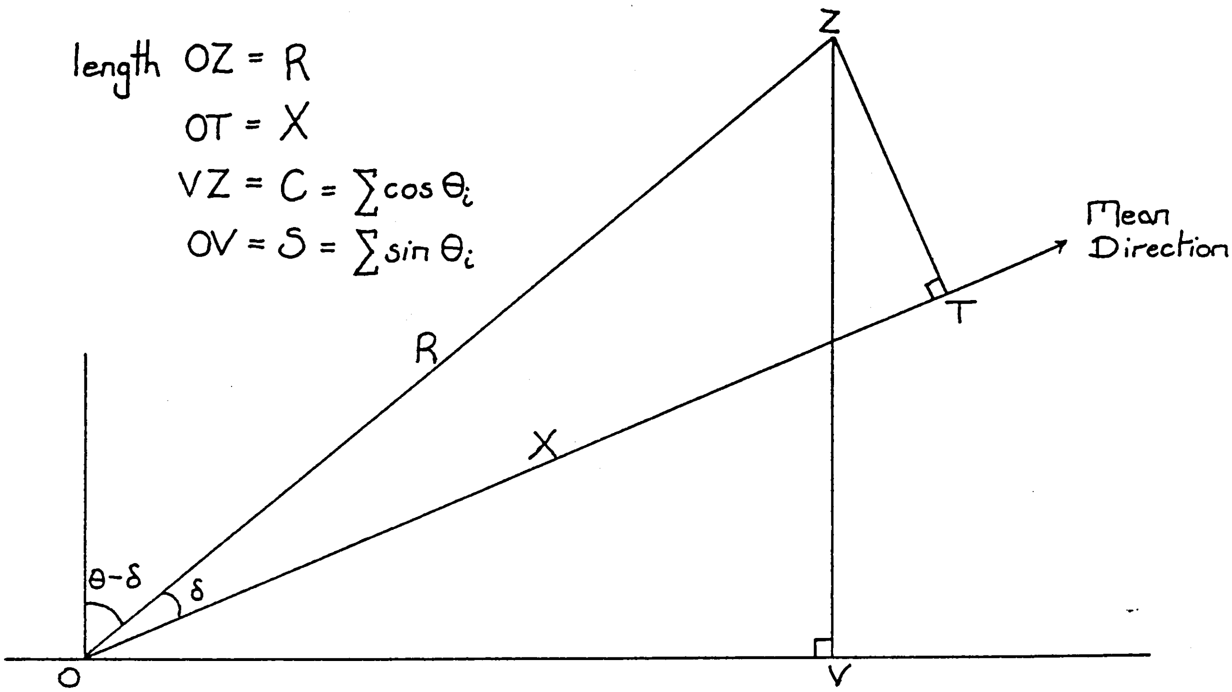


Figure 2.2.1 Statistics R, X, C and S

Clearly from Figure 2.2.1

$$C = R\cos(\theta - \delta)$$

$$S = R\sin(\theta - \delta)$$

$$R^2 = C^2 + S^2$$

The estimates of μ_0 and k by the method of moments are the solutions of

$$\bar{C} = A(k)\cos\mu_0 \qquad \bar{S} = A(k)\sin\mu_0 \tag{2.2.15}$$

which give the same results as the maximum likelihood estimates. Hogg and Craig (1965) have shown that $\bar{\theta}$ and R are jointly complete sufficient statistics for μ_0 and k. However, if k is known then $R\cos\bar{\theta}$ and $R\sin\bar{\theta}$ are minimal sufficient statistics for μ_0 which implies that $\bar{\theta}$ itself does not contain all the information about μ_0.

This underlines the difficulty in constructing an optimal criterion for estimating the circular mean direction. Yet if μ_0 is known then C is a complete sufficient statistic of k, and $\bar{C}$ an unbiased estimate of $A(k)$.

2.3 The Distribution of R for the Uniform and von Mises Distributions

The distribution of R for the Uniform and von Mises distributions were derived and discussed by Stephens (1962a, 62b), Upton (1970) and Mardia (1972) and therefore will not be fully reiterated here. A brief summary of results, however, will be given in order that they may be utilised in later chapters.

2.3.1 Preliminary Results

Using the notation of Mardia (1972)

(a) Let $\psi(\rho,\Phi)$ be the characteristic function of a continuous two-dimensional random variable (x,y) where

$$x = r\cos\theta \qquad y = r\sin\theta \tag{2.3.1}$$

$$\psi(\rho,\Phi) = E[\exp\{i\rho r\cos(\theta - \Phi)\}] \tag{2.3.2}$$

The joint density of r and θ is given by

$$p(r,\theta) = \frac{1}{(2\pi)^2}\, r \int_0^\infty\!\!\int_0^{2\pi} \exp[-i\rho r\cos(\theta - \Phi)]\rho\psi(\rho,\Phi)\quad d\rho d\Phi \tag{2.3.3}$$

Integrating over θ gives the density of r

$$p(r) = \frac{1}{(2\pi)} r \int_0^\infty \int_0^{2\pi} J_0(\rho r) \psi(\rho,\theta) \rho \quad d\rho d\theta \tag{2.3.4}$$

where $J_m(x)$ is the standard Bessel function of order m and real argument.

Equation (2.3.4) is described as an inversion formula for the distribution of r.

Let θ_j,j=1,2...,N be distributed independently with probability density function $f_j(\theta)$, j=1,2,...,N and of unit length and

$$C = \sum_{j=1}^{N} \cos \theta_j \qquad S = \sum_{j=1}^{N} \sin \theta_j$$

The joint characteristic function of (C,S) is given by

$$\prod_{j=1}^{N} \psi_j(\rho,\Phi) \tag{2.3.5}$$

where $\psi_j(\rho,\Phi)$ is the joint characteristic function of $(\cos\theta_j,\sin\theta_j)$.

Hence, from equation (2.3.5) the probability density function of R is given by

$$p(R) = \frac{1}{(2\pi)} R \int_0^\infty \int_0^{2\pi} J_0(\rho R) \; \{\prod_{j=1}^{N} \psi_j(\rho,\Phi)\} \rho \quad d\rho d\Phi \tag{2.3.6}$$

2.3.2 Distribution of R for the Uniform Distribution

To enable the construction of the distribution of R when the observations are taken from the von Mises we shall initially consider the special case of the von Mises when k=0 and the distribution is Uniform.

The problem of finding the density of R is analogous to the problem of random walk. Pearson (1905) required the probability that after N steps a man is at a distance between R and R+δR from his starting point, 0. Here his steps, l, are regarded as of unit length.

Using (1.3.3), the Uniform distribution, in (2.3.2), the characteristic function of $(\cos\theta_j, \sin\theta_j)$ is given by

$$\psi_j(\rho,\Phi) = \frac{1}{(2\pi)} \int_0^{2\pi} \exp[i\rho\cos(\theta - \Phi)] \quad d\theta \tag{2.3.7}$$

From (2.3.5) the c.f. of (C,S) is given by

$$J_0^N(\rho) \tag{2.3.8}$$

Then substituting (2.3.8) into the inversion formula (2.3.6) for R, the probability density function of R for the Uniform distribution is

$$p_u(R) = R \int_0^{\infty} uJ_0(Ru) \; J_0^N(u) \quad du \tag{2.3.9}$$

where

$$p_u(R) = 0 \quad \text{for} \quad R > N$$

The integral (2.3.9) is often referred to as Kluyver's integral (1906). The asymptotic solution of Pearson had been obtained already by Lord Rayleigh (1880). Pearson (1906) gave another proof of Kluyvers result. Rayleigh (1919) used Kluyvers technique to obtain the solution to the problem in three dimensions, or random flights. Tables of $p_u(R)$ for differing N are given by Greenwood and Durand (1955) and updated and extended by Durand and Greenwood (1957). Asymptotic approximations will be discussed in Chapter 4.

2.3.3 Distribution of R for the von Mises Distribution

The joint probability density function of C and S is given by

$$g(C,S) = \frac{1}{2\pi I_0^N(k)} \exp[k\mu C + k\nu S] \frac{p_u(R)}{R} \tag{2.3.10}$$

where

$$\mu = \cos \mu_0 \qquad \nu = \sin \mu_0$$

On transforming C and S to $\bar{\theta}$ and R by $C = R \cos \bar{\theta}$ and $S = R \sin \bar{\theta}$ in (2.3.10) the joint p.d.f. of $\bar{\theta}$ and R is seen to be

$$g(\bar{\theta},R) = \frac{1}{2\pi I_0^N(k)} \exp[kR \cos(\bar{\theta} - \mu_0)]\, p_u(R) \tag{2.3.11}$$

$$0 < \bar{\theta} \leqslant 2\pi \qquad 0 < R < N$$

Integrating with respect to $\bar{\theta}$, the p.d.f. of R for the von Mises is given by

$$p_V(R) = \frac{1}{[I_0(k)]^N} I_0(kR)\, p_u(R) \qquad 0 < R < N \tag{2.3.12}$$

where $p_u(R)$ denotes the p.d.f. for the Uniform distribution given by (2.3.9). Equation (2.3.12) is due to Greenwood and Durand (1955). Asymptotic approximations will be discussed in Chapter 4.

2.4 Combining von Mises Distributions

Let θ_1, θ_2 be independently distributed as von Mises $M(\mu_0, k_1)$, $M(v_0, k_2)$ respectively. The probability distribution function of $\theta = \theta_1 + \theta_2$, using the convolution formula is given by

$$\frac{1}{4\pi^2 I_0(k_1) I_0(k_2)} \int_0^{2\pi} \exp[r \cos(\xi - \beta)]\, d\xi \tag{2.4.1}$$

where

$$\left.\begin{aligned} r \cos \beta &= k_1 + k_2 \cos(\theta - \alpha) \\ r \sin \beta &= k_2 \sin(\theta - \alpha) \\ \alpha &= \mu_0 + v_0 \end{aligned}\right\} \qquad (2.4.2)$$

Proof of Equation (2.4.1), Outlined by Mardia (1972)

The convolution formula is given by

$$\int_0^{2\pi} f_1(\theta) f_2(\zeta - \theta)\, d\theta \qquad (2.4.3)$$

$$= \int_0^{2\pi} \frac{1}{2\pi I_0(k_1)} \exp[k_1 \cos(\theta - \mu_0)] \frac{1}{2\pi I_0(k_2)} \exp[k_2 \cos(\zeta - (\theta - v_0))]$$

$$= \frac{1}{4\pi^2 I_0(k_1) I_0(k_2)} \int_0^{2\pi} \exp[k_1 \cos(\theta - \mu_0) + k_2 \cos(\zeta - (\theta - v_0))]\, d\theta \qquad (2.4.4)$$

Taking the exponential term of the integral

$$k_1 \cos(\theta - \mu_0) + k_2 \cos(\zeta - (\theta - v_0)) \qquad (2.4.5)$$

Let $\xi = \theta - \mu_0$

Then (2.4.4) becomes

$$k_1 \cos \xi + k_2 \cos(\zeta - (\xi + \mu_0 - v_0)) \qquad (2.4.6)$$

Expanding and using the sine and cosine rules produces

$$[k_1 + k_2 \cos(\zeta - \mu_0 - v_0)] \cos \xi + [k_2 \sin(\zeta - \mu_0 - v_0)] \sin \xi \qquad (2.4.7)$$

Using the equalities (2.4.2)

$$= [r \cos \beta] \cos \xi + [r \sin \beta] \sin \xi$$

$$= r \cos(\xi - \beta) \qquad (2.4.8)$$

Replacing (2.4.7) into (2.4.4) gives equation (2.4.1)

$$\text{As} \quad \int_0^{2\pi} \exp[k \cos \theta] \, d\theta = 2\pi I_0(k) \tag{2.4.9}$$

(2.4.1) may be reduced i.e.

$$\int_0^{2\pi} \exp[k \cos \theta] \, d\theta = \int_0^{2\pi} \exp[r \cos(\xi - \beta)] \, d\xi$$

where k=r

$$r = [[k_1 + k_2 \cos(\theta - \alpha)]^2 + [k_2 \sin(\theta - \alpha)]^2]^{\frac{1}{2}}$$

$$= k_1^2 + k_2^2 + 2k_1k_2 \cos(\theta - \alpha) \tag{2.4.10}$$

From (2.4.9), (2.4.4) now becomes

$$\frac{1}{4\pi^2 I_0(k_1) I_0(k_2)} \; 2\pi I_0(k)$$

$$= \frac{1}{2\pi I_0(k_1) I_0(k_2)} \; I_0[(k_1^2 + k_2^2 + 2k_1k_2 \cos(\theta - \alpha))^{\frac{1}{2}}] \tag{2.4.11}$$

If k_1=k_2=k, and using $\cos 2\theta = 2\cos^2\theta - 1$, (2.4.11) becomes

$$\frac{1}{2\pi I_0^2(k)} \; I_0[2k \cos \tfrac{1}{2}(\theta - \alpha)] \tag{2.4.12}$$

The expression (2.4.12) is not the density of a von Mises distribution, i.e. the convolution of von Mises distributions is not a von Mises distribution. However, expression (2.4.12) may be approximated by a von Mises distribution. Without loss of generality, let μ_0 and v_0 equal zero, the distributions M(0,k_1) and M(0,k_2) may now be approximated by the wrapped Normal distribution. The wrapped Normal, as discussed in Section 1.3, holds the additive property, therefore two wrapped Normals gives another wrapped Normal with parameter $\sigma_3^2 = \sigma_1^2 + \sigma_2^2$. This distribution can then be approximated by M(0,k_3) where k_3 is the solution of

$$A(k_3) = A(k_1)A(k_2) \tag{2.4.13}$$

Hence

$$\left.\begin{array}{l} (\theta_1 + \theta_2) \bmod 2\pi \text{ is approximately } M(\mu_0 + v_0, k_3) \\ \text{and} \\ (\theta_1 - \theta_2) \bmod 2\pi \text{ is approximately } M(\mu_0 - v_0, k_3) \end{array}\right\} \quad (2.4.14)$$

Stephens (1963) has shown numerically that this approximation is satisfactory, although reducing in accuracy as k decreases.

2.5 A Summary of Exact and Approximate Moments of R

Full details of the exact and approximate moments of R may be found in Upton (1970) and Mardia (1972). Here only those necessary for the improvement and examination of tests discussed in later chapters will be given.

The distribution (2.3.12) cannot be used directly to obtain the expected values of R, however, for large or small values of k (2.3.12) may be replaced by approximation expressions from which the expectation of R for differing k may be calculated.

Stephens (1969) was the first to suggest and undertake the method of repeated differentiation of the probability density function to obtain the exact even moments of R. Upton (1970) having defined S and C by

$$S = \sum_{i=1}^{N} \sin \theta_i \qquad C = \sum_{i=1}^{N} \cos \theta_i$$

where $\quad R^2 = S^2 + C^2 \qquad (2.5.1)$

utilised the moment generating functions of S and C to obtain their exact expectations and in turn produce the expectation of R^2 as did Stephens, as

$$E(R^2) = N + N(N - 1)\rho^2 \qquad (2.5.2)$$

where

$$\rho = A(k) = \frac{I_1(k)}{I_0(k)}$$

As $k \to 0$, then $\rho \to 0$ and $E(R^2) \to N$, a result expected for the Uniform distribution. As the distribution becomes more concentrated about its angular mean direction ($k \to \infty$) then $\rho \to 1$ and $E(R^2) \longrightarrow N^2$.

Using both these approaches the exact expectation of R^4 has been calculated and given as

$$E(R^4) = \frac{N!}{(N-4)!}\rho^4 + \frac{2N!}{(N-3)!}\rho^2(2+\rho_2)$$

$$+ \frac{N!}{(N-2)!}(2 + 4\rho^2 + \rho_2^2) + N \qquad (2.5.3)$$

where

$$\rho_2 = \frac{I_2(k)}{I_0(k)} = 1 - 2\left[\frac{A(k)}{k}\right]$$

Upton (1970) gives several approximations to the expectation of R by equating distribution approximations given by Watson and Williams (1956) and Stephens (1969) to their associated expected chi-squared values for large and small values of k. The expectation of R for the von Mises distribution, as $N \longrightarrow \infty$, is given by

$$E(R) = N\rho + \frac{1}{4\rho}(1 - \rho^2) + 0\left[\frac{1}{N}\right]$$

$$\approx N\rho + \frac{1}{2k} \qquad (2.5.4)$$

From Watson and Williams approximation, for large k

$$E(R) \approx N - \frac{(N-1)}{2k} - \frac{(N^2-1)}{8Nk^2} + o\left[\frac{N}{k^2}\right] \qquad (2.5.5)$$

From Stephens approximation, for large k

$$E(R) \approx N - \frac{(N-1)}{2k} - \frac{3(N-1)}{16k^2} \qquad (2.5.6)$$

If k and N are large, by substituting the Bessel functions, discussed later in Chapter 3;

$$E(R) \approx N - \frac{(N - 1)}{2k} - \frac{N}{8k^2} \tag{2.5.7}$$

which is in close agreement with the result (2.5.5).

If k is small, without being too small for the approximation to the ratio of Bessel function to be invalid.

$$E(R) \approx \frac{1}{2k} + \frac{Nk}{2} \tag{2.5.8}$$

CHAPTER 3

LIKELIHOOD RATIO TESTS AND APPROXIMATIONS TO THE MAXIMUM LIKELIHOOD ESTIMATES

3.1 The Method of Maximum Likelihood and Likelihood Ratio Tests

In the early 1920's, R A Fisher proposed a general method of estimation, called the method of maximum likelihood. Fisher demonstrated the advantage of this method by showing that (1) it yields sufficient estimates whenever they exist, and (2) it yields estimates which are asymptotically (when $N \to \infty$) minimum variance unbiased estimators. In principle, the method of maximum likelihood consists of selecting that value of the parameter θ under consideration for which $f(x_1, x_2, \ldots, x_n; \theta)$, the probability of obtaining the sample values, is a maximum.

The joint likelihood of the N observations $\theta_1, \theta_2, \ldots, \theta_N$ from a von Mises distribution with parameters k and μ_0 is

$$L(k,\mu_0) = \left[\frac{1}{2\pi I_0(k)}\right]^N \exp\{k \sum_{i=1}^{N} \cos(\theta_i - \mu_0)\} \tag{3.1.1}$$

as was given and used in Chapter 2.2.

Likelihood ratio tests utilise maximum likelihood estimates to test whether a particular set of data is consistent with some hypothesis about its underlying distribution. The likelihood ratio test is a uniformly most powerful test. A detailed discussion of these tests originally formulated by Neyman and Pearson, can be seen in Kendall and Stuart (1967).

The likelihood ratio test provides a means by which a null hypothesis can be tested against an alternative hypothesis. A null hypothesis $k = \hat{k}_0$, $\mu_0 = \hat{\mu}_0$ may be tested against an alternative hypothesis $k = \hat{k}_1$, $\mu_0 = \hat{\mu}_0$ parameters of the population given by $f(\theta;\ k,\ \mu_0)$. Let L_0 and L_1 denote the likelihoods of k_0, μ_0 and k_1, μ_1 given the population with its parameters k and μ_0. Symbolically,

$$L_0 = \prod_{i=1}^{N} f(\theta_i;\ \hat{k}_0,\ \hat{\mu}_0) \quad \text{and} \quad L_1 = \prod_{i=1}^{N} f(\theta_i;\ \hat{k}_1,\ \hat{\mu}_1) \tag{3.1.2}$$

These quantities are both values of random variables, they depend on the observed sample values θ_1, θ_2,... θ_N, and their ratio.

$$\lambda = \frac{\max\ \{L_0\}}{\max\ \{L_1\}} : \frac{\text{under null hypothesis}}{\text{under alternative hypothesis}} \tag{3.1.3}$$

which is referred to as a value of the likelihood ratio statistic λ.

Since max L_0 is apt to be small compared to max L_1 when the null hypothesis is false, then the null hypothesis should be rejected when λ is small.

Usually the natural logarithm of the ratio (3.1.3) is taken since, for large N, the distribution of $-2\log\ \lambda$ approaches, under very general conditions, the chi–squared distribution with its degrees of freedom given by the number of parameters which are constrained by the null hypothesis. Let, under the null hypothesis, the best estimates of k and μ_0 be $\hat{k}_0$ and $\hat{\mu}_0$ respectively, where these are either given values specified by the hypothesis or the maximum likelihood estimates of the parameters under the null hypothesis.

Similarly, let $\hat{k}_1$ and $\hat{\mu}_1$ be the corresponding estimates under the alternative hypothesis. Then, using (3.1.3) the likelihood ratio statistic is

$$\lambda = \frac{L_0(\hat{k}_0, \hat{\mu}_0)}{L_1(\hat{k}_1, \hat{\mu}_1)} \tag{3.1.4}$$

For observations from the von Mises distribution

$$\lambda = \left[\frac{I_0(\hat{k}_1)}{I_0(\hat{k}_0)}\right]^N \exp\{\hat{k}_0\sum_{i=1}^{N} \cos(\theta_i - \hat{\mu}_0) - \hat{k}_1\sum_{i=1}^{N} \cos(\theta_i - \hat{\mu}_1)\} \tag{3.1.5}$$

It is, therefore, not necessary to rely on being able to derive the distribution of our test statistic theoretically since a good approximation to the distribution, using $-2\log \lambda$, is available.

Since this, and as we will see later, several other successive approximations to the likelihood ratio test statistics are used, invariably the tests are biased. However, by equating the expectation of the test statistic to its associated chi-square expectation (given by its degrees of freedom), we may attempt to remove this bias, and hence obtain a more effective test.

Upton (1973, 76) utilizes this method extensively to improve his statistics for single-sample and multi-sample tests of the von Mises distribution. (These tests will be discussed and summarised in Chapter 4.) Many of the test statistics resulting from the likelihood ratio method could be used without simplication. By using the various approximations, test statistics which are simpler, both in form and use, may be derived.

3.2 Approximations for the von Mises Concentration Statistic, $\hat{k}$, when k is Small

In Chapter 2, we have seen that if $\theta_1,\ldots,\theta_N$ are a random sample from $M(\mu_0,k)$ then the maximum likelihood estimate of k, $\hat{k}$, is the solution of

$$A(\hat{k}) = \frac{I_1(\hat{k})}{I_0(\hat{k})} = \frac{R}{N} = (\bar{C}^2 + \bar{S}^2)^{\frac{1}{2}} \tag{3.2.1}$$

thus

$$\hat{k} = A^{-1}\left[\frac{R}{N}\right] \tag{3.2.2}$$

where the ratio of the Bessel functions $I_1(k)/I_0(k)$ will be denoted as $A(k)$.

Limited tables of A^{-1} are given by Mardia (1972, p 298) and Batschelet (1980, Tables B, C and D) based on those in Gumbel, Greenwood and Durand (1953). Mardia and Zemroch (1975) gave a computer algorithm for calculating $\hat{k}$ and other circular statistics by an iterative process.

In this and section 3.3 several approximations to A^{-1}, which do not need tables or large computing equipment, are given. From these functions the statistic $\hat{k}$ can be obtained fairly accurately. The approximations stated here are taken from Dobson (1978) and Upton (1970). In a similar manner to Upton we shall denote R/N and X/N by x. This causes no problems since the choice is determined by whether or not the mean direction is known or not.

Dobson initially states four approximations to A^{-1} which are global approximations for all values of k. The first uses Amos (1974) equation that

$$\frac{x}{\frac{1}{2} + \left[x^2 + \frac{9}{4}\right]^{\frac{1}{2}}} < A(x) < \frac{x}{\frac{1}{2} + \left[x^2 + \frac{1}{4}\right]^{\frac{1}{2}}} \tag{3.2.3}$$

and hence that $A^{-1}(x)$ is approximately

$$A_1^{-1}(x) = \left\{\frac{x}{1 - x^2}\right\}\left\{\frac{1}{2} + \left[c(1 - x^2) + \frac{1}{4}\right]^{\frac{1}{2}}\right\} = \hat{k}_1 \qquad (3.2.4)$$

where c = 1.46 to minimize the maximum relative error, and so k can be estimated using $\hat{k} = A_1^{-1}(R/N)$.

The other three global approximations use a noticed feature that A(x) behaves like $(2/\pi)\tan^{-1}x$ and so $A^{-1}(x)$ is like $\tan(\pi x/2)$. From this Dobson states the approximation

$$A_2^{-1}(x) = \left[\frac{4}{\pi} + x^2\left\{\frac{\pi}{4} - \frac{4}{\pi}\right\}\right]\tan\left[\frac{\pi x}{2}\right] = \hat{k}_2 \qquad (3.2.5)$$

Improvements are found by replacing terms in (3.2.5) by minimax values, giving

$$A_3^{-1}(x) = \left[1.32 + x^2\left\{\frac{1}{1.32} - 1.32\right\}\right]\tan\left[\frac{\pi x}{2}\right] = \hat{k}_3 \qquad (3.2.6)$$

and

$$A_4^{-1}(x) = (1.28 - 0.53x^2)\tan\left[\frac{\pi x}{2}\right] = \hat{k}_4 \qquad (3.2.7)$$

Compared to these global approximations, approximations to $A^{-1}(x)$ for particular parts of its range were studied by Upton. The power series for the Bessel functions $I_0(x)$ and $I_1(x)$ for small x gives

$$A(\hat{k}) = \frac{1}{2}\hat{k}\left[1 - \frac{\hat{k}^2}{8} + \frac{\hat{k}^4}{48} + O(\hat{k}^6)\right] \qquad (3.2.8)$$

On inverting the series and taking the first three terms an approximation of $A^{-1}(x)$ is given by

$$A_5^{-1}(x) = x\left[2 + x^2 + \frac{5x^4}{6}\right] = \hat{k}_5 \qquad (3.2.9)$$

Using the first two terms

$$A_6^{-1}(x) = x[2 + x^2] = \hat{k}_6 \tag{3.2.10}$$

Finally if k is expected to be very small, then we may simply use the first term approximation of the power series, which gives

$$A_7^{-1}(x) = 2x = \hat{k}_7 \tag{3.2.11}$$

To examine these approximations further Table 3.1 lists the true value of k against the approximations $\hat{k}_1$ to $\hat{k}_7$ between 0 and 1. For quicker and easier appreciation of the accuracy of the approximations Figures 3.1(a) to 3.1(g) plot the residuals, $k-\hat{k}_i$, against the true k. Figures 3.2(a) to 3.2(g) plot the relative percentage errors, $|k-\hat{k}_i|/k$, for each of the approximations.

From the global approximations, $\hat{k}_1,\ldots,\hat{k}_4$, for values of k less than 1, $\hat{k}_2$ and $\hat{k}_4$ are clearly the best, with maximum relative errors, illustrated in Figures 3.2(b) and 3.2(d), of 0.71% and 0.84%, respectively at k=1.

By far the best approximation obtained from the power series is $\hat{k}_5$, for all values of $k < 1$, with maximum relative error of 0.35%. For very small values of k (less than 0.2 or R/N less than 0.1), however, $\hat{k}_7$ is by far the simplest and quickest method of estimating. From Figures 3.1(g) and 3.2(g) we can see that, outside the range of $0 < k < 0.2$, $\hat{k}_7$ deteriorates rapidly.

A disappointing function approximation is $\hat{k}_1$ with maximum relative error of 9.61% for k=0.05 (the minimum value of k tested). For values of x (ie R/N) near to zero, the global approximations $\hat{k}_1,\ldots,\hat{k}_4$ are not as good as those obtained from the power series.

TABLE 3.1

APPROXIMATIONS TO SMALL K

True k	x	$\hat{k}_1$	$\hat{k}_2$	$\hat{k}_3$	$\hat{k}_4$	$\hat{k}_5$	$\hat{k}_6$	$\hat{k}_7$
0	0	0	0	0	0	0	0	0
0.05	0.0250	0.0452	0.0500	0.0518	0.0503	0.0500	0.0500	0.0500
0.10	0.0499	0.0904	0.1000	0.1036	0.1005	0.1000	0.1000	0.0999
0.15	0.0748	0.1357	0.1499	0.1554	0.1507	0.1500	0.1500	0.1496
0.20	0.0995	0.1811	0.1999	0.2071	0.2009	0.2000	0.2000	0.1990
0.25	0.1240	0.2266	0.2498	0.2588	0.2510	0.2500	0.2500	0.2481
0.30	0.1483	0.2723	0.2996	0.3103	0.3010	0.3000	0.2999	0.2967
0.35	0.1724	0.3182	0.3494	0.3618	0.3509	0.3500	0.3499	0.3447
0.40	0.1961	0.3643	0.3991	0.4131	0.4008	0.4000	0.3997	0.3922
0.45	0.2195	0.4106	0.4488	0.4643	0.4505	0.4500	0.4496	0.4390
0.50	0.2425	0.4572	0.4984	0.5154	0.5001	0.5000	0.4993	0.4850
0.55	0.2651	0.5041	0.5480	0.5663	0.5496	0.5499	0.5488	0.5302
0.60	0.2873	0.5513	0.5975	0.6171	0.5991	0.5999	0.5982	0.5742
0.65	0.3090	0.5989	0.6469	0.6678	0.6484	0.6498	0.6474	0.6179
0.70	0.3302	0.6468	0.6964	0.7184	0.6976	0.6996	0.6963	0.6604
0.75	0.3509	0.6951	0.7458	0.7689	0.7467	0.7494	0.7450	0.7018
0.80	0.3711	0.7438	0.7952	0.8192	0.7958	0.7991	0.7932	0.7421
0.85	0.3907	0.7929	0.8446	0.8695	0.8448	0.8487	0.8411	0.7815
0.90	0.4098	0.8424	0.8940	0.9197	0.8937	0.8981	0.8885	0.8197
0.95	0.4284	0.8924	0.9434	0.9698	0.9427	0.9474	0.9354	0.8568
1.00	0.4464	0.9428	0.9929	1.0199	0.9916	0.9965	0.9817	0.8928

approximations for small k against k.

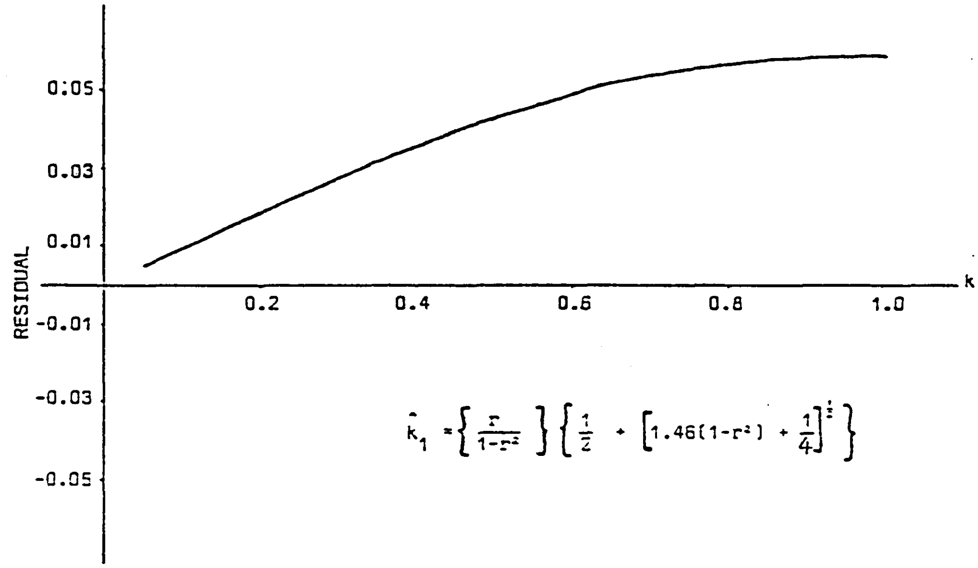


Figure 3.1(a)

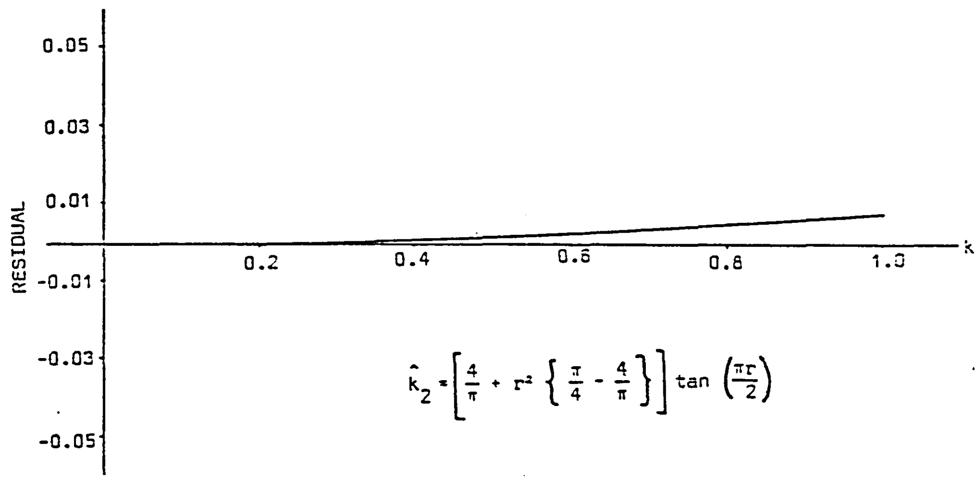


Figure 3.1(b)

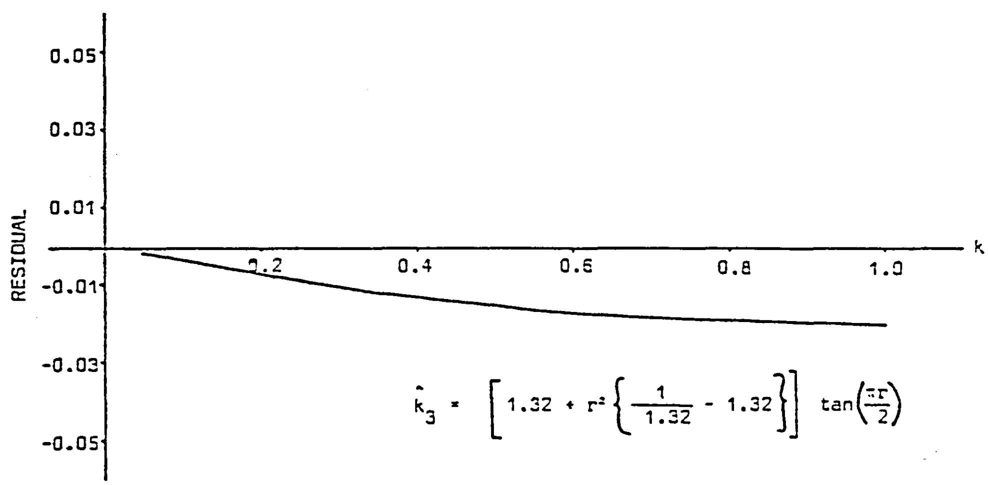


Figure 3.1(c)

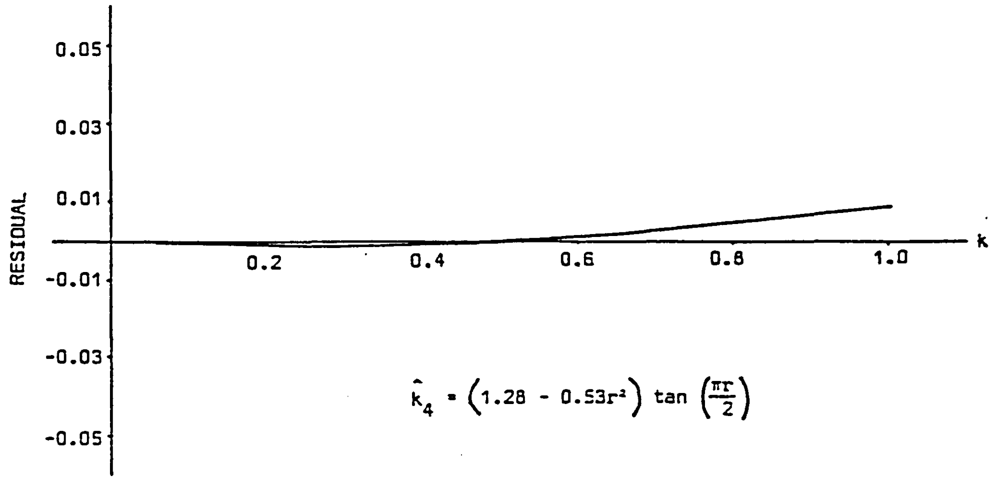


Figure 3.1(d)

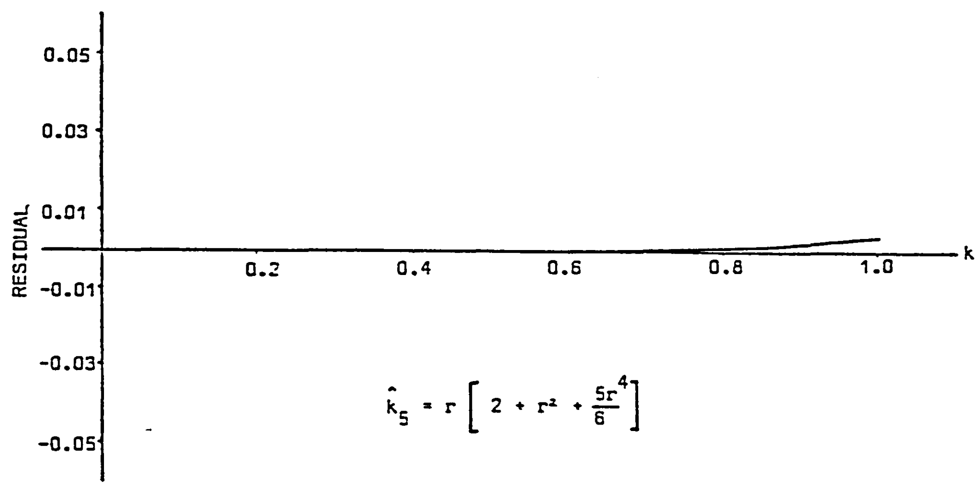


Figure 3.1(e)

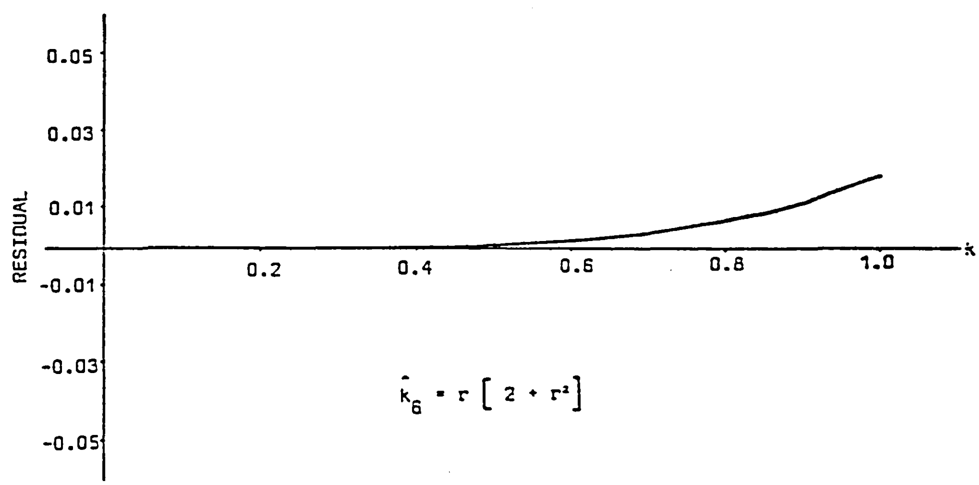


Figure 3.1(f)

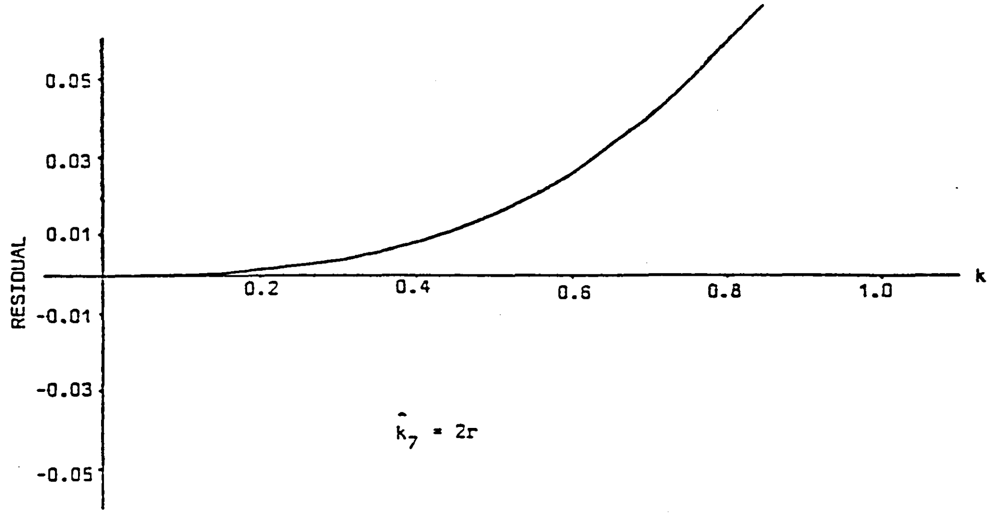


Figure 3.1(g)

Figures 3.2(a) to (g). Plots of the relative percentage error of the approximations for small k against k.

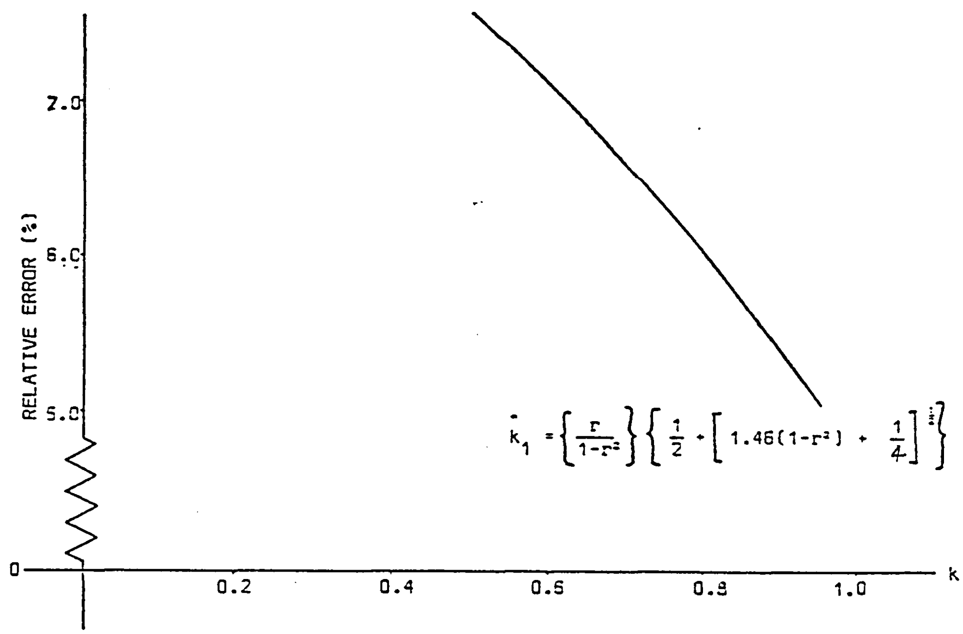


Figure 3.2(a)

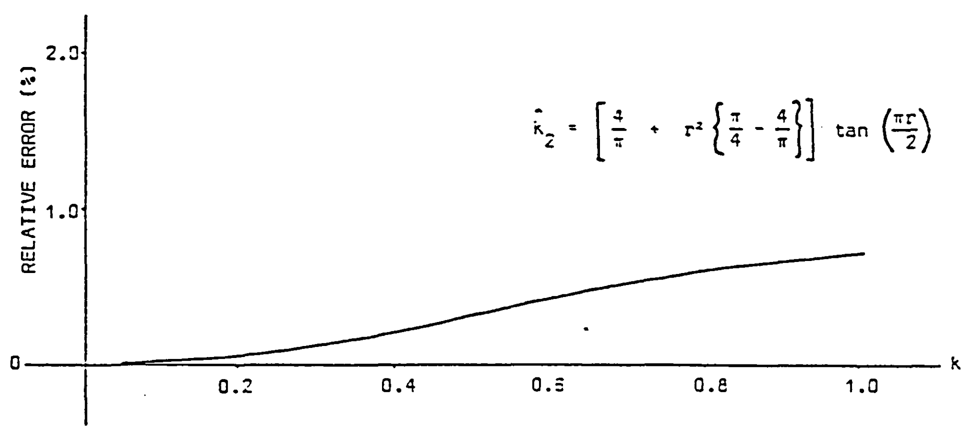


Figure 3.2(b)

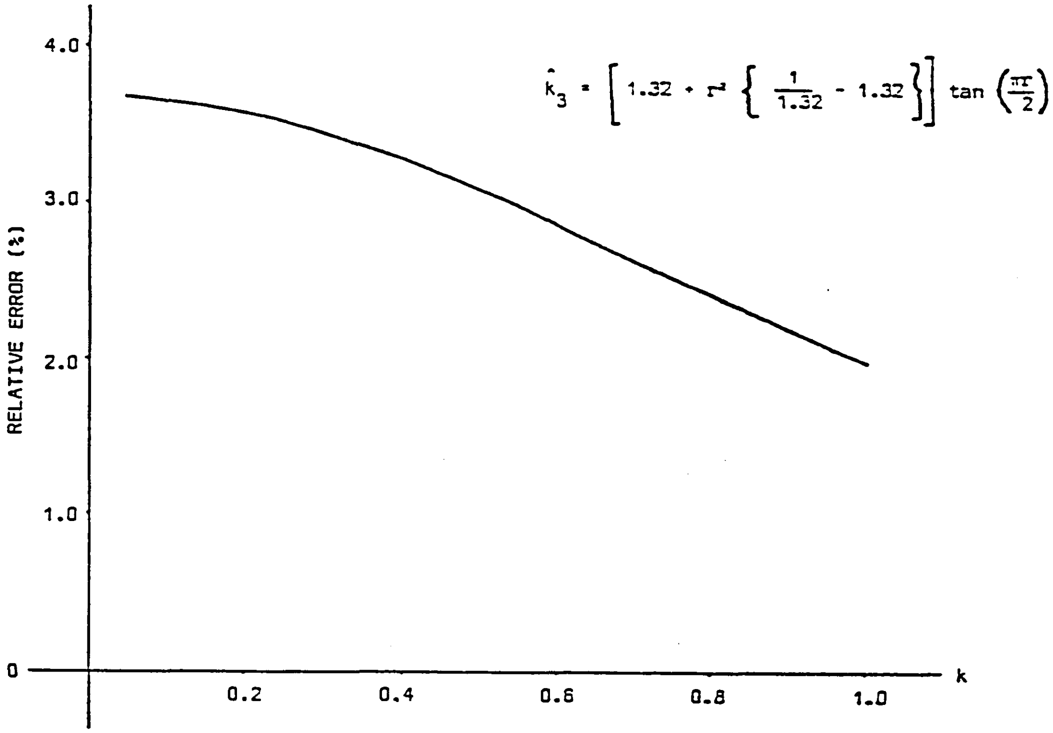


Figure 3.2(c)

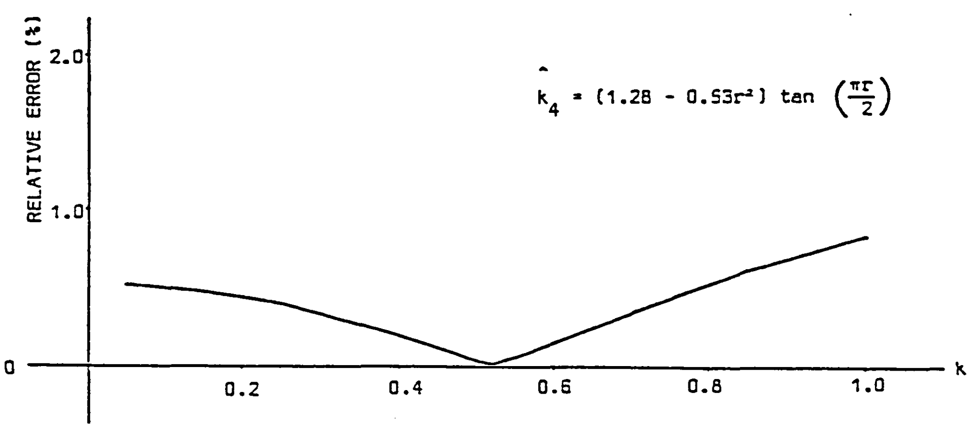


Figure 3.2(d)

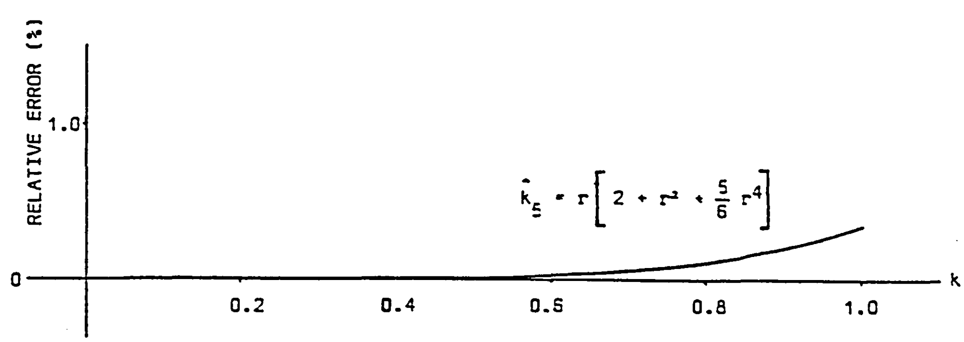


Figure 3.2(e)

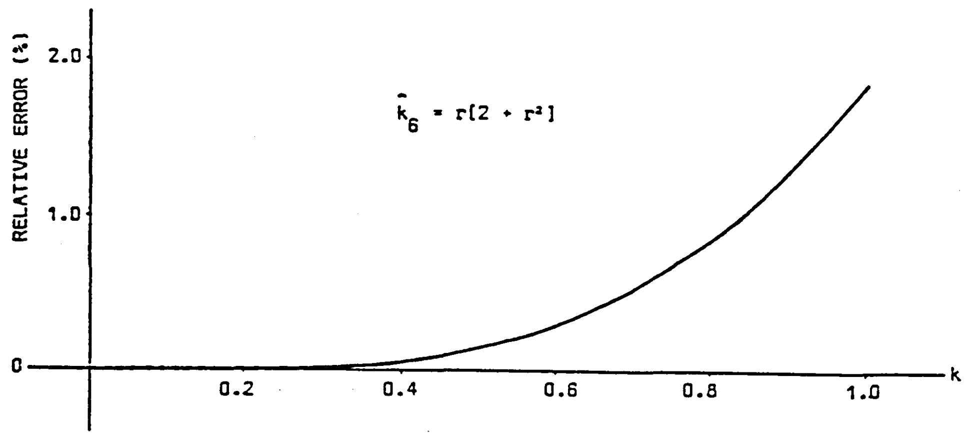


Figure 3.2(f)

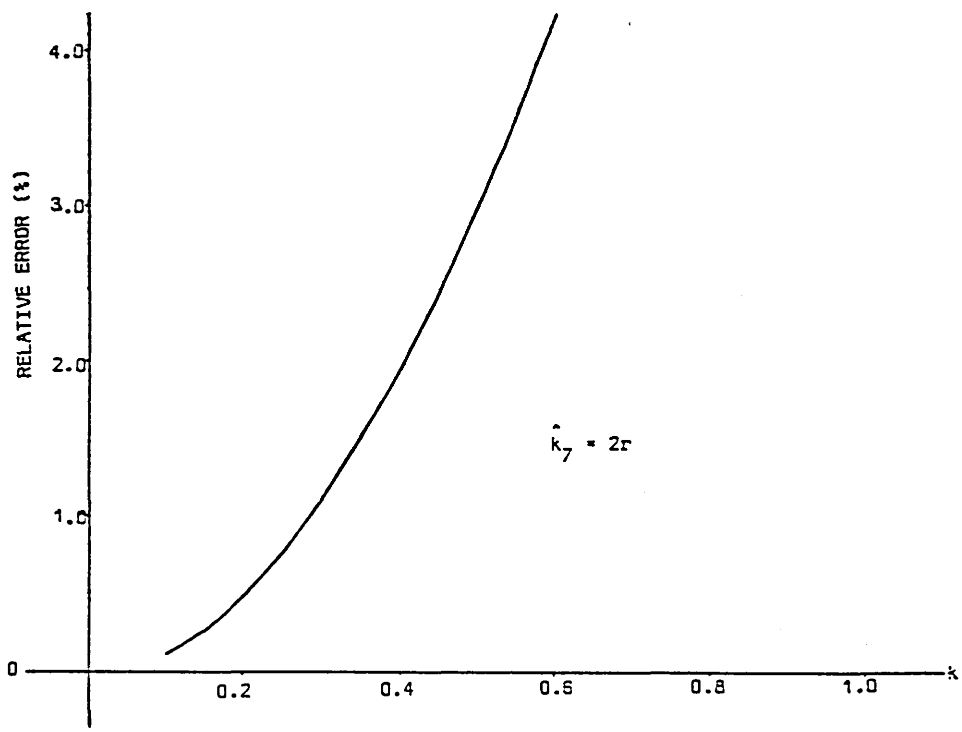


Figure 3.2(g)

3.3 Approximations for the von Mises Concentration Statistic, $\hat{k}$, when k is Large

For large values of k the power series expansion of Bessel functions does not provide a simple approximation to (3.2.1), however, the asymptotic expansion quoted by Abramowitz and Stegun (1965) may be used

$$I_p(k) \approx \frac{\exp[k]}{(2k)^{\frac{1}{2}}}\left\{1 - \frac{m-1}{8k} + \frac{(m-1)(m-9)}{2!\ (8k)^2} - \frac{(m-1)(m-9)(m-25)}{3!\ (8k)^3} \ldots\right\} \tag{3.3.1}$$

where $m = 4p^2$. Using this relation

$$A(\hat{k}) \approx 1 - \frac{1}{2\hat{k}} - \frac{1}{8\hat{k}^2} - \frac{1}{8\hat{k}^3} + 0(\hat{k}^{-4}) \tag{3.3.2}$$

Denoting R/N and X/N by x, using Maclaurins theorem the solution of (3.3.2) is

$$\frac{1}{\hat{k}} = 2(1-x) - (1-x)^2 - (1-x)^3 \tag{3.3.3}$$

On inverting we obtain, as the first three terms

$$\hat{k} = \frac{1}{2(1-x)} + \frac{1}{4} + \frac{3}{8}(1+x) \tag{3.3.4}$$

$$A_8^{-1}(x) = \frac{9 - 8x - 3x^2}{8(1-x)} = \hat{k}_8 \tag{3.3.5}$$

Using the first two terms of (3.3.3) only

$$A_9^{-1}(x) = \frac{(3-x)}{4(1-x)} = \hat{k}_9 \tag{3.3.6}$$

If k is expected to be very large, then we may simply use the first term approximation of the power series, giving

$$A_{10}^{-1}(x) = \frac{1}{2(1-x)} = \hat{k}_{10} \tag{3.3.7}$$

Upton (1970) gives his solution of (3.3.2) as

$$\frac{1}{\hat{k}} = 2(1 - x) - (1 - x)^2 - \frac{3}{2}(1 - x)^3 \tag{3.3.8}$$

On inverting

$$\hat{k} = \frac{1}{2(1 - x)} + \frac{1}{4} + \frac{(1 - x)}{2} \tag{3.3.9}$$

$$A_{11}^{-1}(x) = \frac{5 - 5x + 2x^2}{4(1 - x)} = \hat{k}_{11} \tag{3.3.10}$$

Upton does not explain how he derives equation (3.3.8) and therefore does not state why the third term is different to that of equation (3.3.3), however, as we shall see later in the chapter, the approximation $\hat{k}_{11}$ is better than $\hat{k}_8$.

Upton (1973) states an approximation suggested by M A Stephens where the variance of the maximum likelihood estimator is considered, giving

$$A_{12}^{-1}(x) = \frac{1}{1 - x^2} = \hat{k}_{12} \tag{3.3.11}$$

To examine the approximations $\hat{k}_8, \ldots \hat{k}_{12}$ and the global approximations $\hat{k}_1, \ldots \hat{k}_4$, given in Section 3.2 for large values of k, Table 3.2 lists the true values of k between 1 and 20 against these approximations.

As with small values of k, the residuals, $k-\hat{k}_i$, have been plotted for each approximation for easier appreciation of their accuracy, given in Figures 3.3(a) to 3.3(i). Similarly Figures 3.4(a) to 3.4(i) plot the relative percentage errors for each of the approximations.

Figures 3.3(a) to 3.3(d) show the residuals for the global approximations, $\hat{k}_1, \ldots \hat{k}_4$. Due to the large residuals shown within these approximations the graph scales, for analysis of large k, have been increased compared to those of small k (Figures 3.1 (a) to 3.1(g)). Although these scales have increased, $\hat{k}_1$ equation (3.2.4), has such large residuals and relative errors, compared to the other approximations, that the plots 3.3(a) and 3.4(a) still leave the graph. A residual of -0.8183 would be seen at k=20, and a maximum relative error of 9.6 at approximately k=4.5. Function approximation, $\hat{k}_1$, is unexpectedly poor for both small and large k especially considering its complex form.

From the remaining global approximations, $\hat{k}_3$ has the smallest maximum actual residual, in the range k=1 to 20, of 0.2634 at k=20, in comparison $\hat{k}_4$ has maximum actual residual of 0.4944 at k=20. However, $\hat{k}_4$ may be seen as the best global function approximation since it has a maximum relative error of 2.47% at k=20, while $\hat{k}_3$ has a maximum relative error of 3.42 at k=3.85.

Of the five power series functions, from examination of Figures 3.3(e) to 3.3(i), $\hat{k}_8$ and $\hat{k}_{11}$ are the better two approximations, with maximum residuals of 0.048 at k=3.1, and 0.025 at k=3.2, respectively, for values of $k_i \geqslant 2.5$. On examination of the relative percentage errors $\hat{k}_{11}$ is the best approximation with maximum relative error of 0.79% at k=3.1, compared to 1.61% at k=2.85 for $\hat{k}_8$, for values of $k_1 \geqslant 2.5$.

It is interesting to note the adequacy of $\hat{k}_{10}$, equation (3.3.7), as this is often quoted as a good approximation for large k. Form Figure 3.3(g) we can see that a far greater improvement would be seen if 0.25 was added to the approximation, producing $\hat{k}_9$.

For almost all $\hat{k}_1$, ..., $\hat{k}_{12}$ approximations the actual and relative errors fluctuate most in the range $1 < k < 2.5$.

TABLE 3.2

APPROXIMATIONS TO LARGE K

True k	x	$\hat{k}_1$	$\hat{k}_2$	$\hat{k}_3$	$\hat{k}_4$	$\hat{k}_8$	$\hat{k}_9$	$\hat{k}_{10}$	$\hat{k}_{11}$	$\hat{k}_{12}$
1.0	0.4464	0.9428	0.9929	1.0199	0.9916	1.3608	1.1532	0.9032	1.4300	1.2489
1.5	0.5961	1.4717	1.4947	1.5222	1.4835	1.6395	1.4880	1.2380	1.6900	1.5513
2.0	0.6978	2.0392	2.0153	2.0356	1.9885	2.0177	1.9044	1.6544	2.0555	1.9489
2.5	0.7650	2.6281	2.5531	2.5612	2.5068	2.4658	2.3776	2.1276	2.4951	2.4109
3.0	0.8100	3.2200	3.0981	3.0910	3.0302	2.9526	2.8814	2.6314	2.9764	2.9076
3.5	0.8411	3.8034	3.6409	3.6174	3.5505	3.4563	3.3967	3,1467	3.4762	3,4183
4.0	0.8635	4,3740	4.1772	4.1365	1.0638	3.9648	3.9136	3.6636	3.9818	3.9319
4.5	0.8803	4.9319	4.7059	4.6479	4.5697	4.4731	4.4282	4.1782	4.4880	4.4441
5.0	0.8934	5.4791	5.2282	5.1527	5.0692	4.9796	4.9396	4.6897	4.9930	4.9537
5.5	0.9038	6.0180	5.7455	5.6525	5.5637	5.4845	5.4484	5.1984	5.4965	5,4610
6.0	0.9124	6.5506	6.2592	6.1486	6.0547	5.9880	5.9551	5.7051	5.9989	5.9666
6.5	0.9195	7.0781	6.7700	6.6418	6.5428	6.4904	6.4902	6.2102	6.5005	6.4707
7.0	0.9255	7.6020	7.2789	7.1332	7.0291	6.9923	6.9642	6.7143	7.0016	6.9740
7.5	0.9307	8.1227	7.7863	7.6229	7.5139	7.4925	7.4675	7.2175	7.5022	7.4765
8.0	0.9352	8.6410	8.2924	8.1115	7.9975	7.9945	7.9702	7.7202	8.0026	7.9786
8.5	0.9392	9.1572	8.7978	8.5992	8.4802	8.4952	8.4724	8.2224	8.5028	8.4803
9.0	0.9427	9.6720	9.3025	0.0863	8.9624	8.9960	8.9745	8.7245	9.0031	8.9818
9.5	0.9458	10.1852	9.8066	9.5727	9.4439	9.4965	9.4762	9.2262	9.5033	9.4831
10.0	0.9486	10.6972	10.3102	10.0587	9.9249	9.9968	9.9776	9.7276	10.0033	9.9841
11.0	0.9534	11.7180	11.3161	11.0293	10.8857	10.9973	10.9798	10.7298	11.0031	10.9859
12.0	0.9574	12.7359	12.3212	11.9990	11.8457	11.9978	11.9818	11.7318	12.0031	11.9873
13.0	0.9607	13.7514	13.3254	12.9679	12.8048	12.9983	12.9836	12.7336	13.0032	12.9886
14.0	0.9636	14.7647	14.3289	13.9360	13.7631	13.9985	13.9849	13.7349	14.0031	13.9895
15.0	0.9661	15.7766	15.3319	14.9037	14.7211	14.9989	14.9862	14.7362	15.0032	14.9905
16.0	0.9682	16.7867	16.3343	15.8707	15.6784	15.9988	15.9869	15.7369	16.0028	15.9909
17.0	0.9701	17.7959	17.3364	16.8374	16.6355	16.9989	16.9877	16.7377	17.0027	16.9915
18.0	0.9718	18.8044	18.3388	17.8044	17.5929	17.9993	17.9887	17.7387	18.0028	17.9922
19.0	0.9733	19.8114	19.3401	18.7703	18.5490	18.9990	18.9890	18.7390	19.0024	18.9924
20.0	0.9747	20.8182	20.3418	19.7366	19.5056	19.9990	19.9895	19.7395	20.0021	19.9927

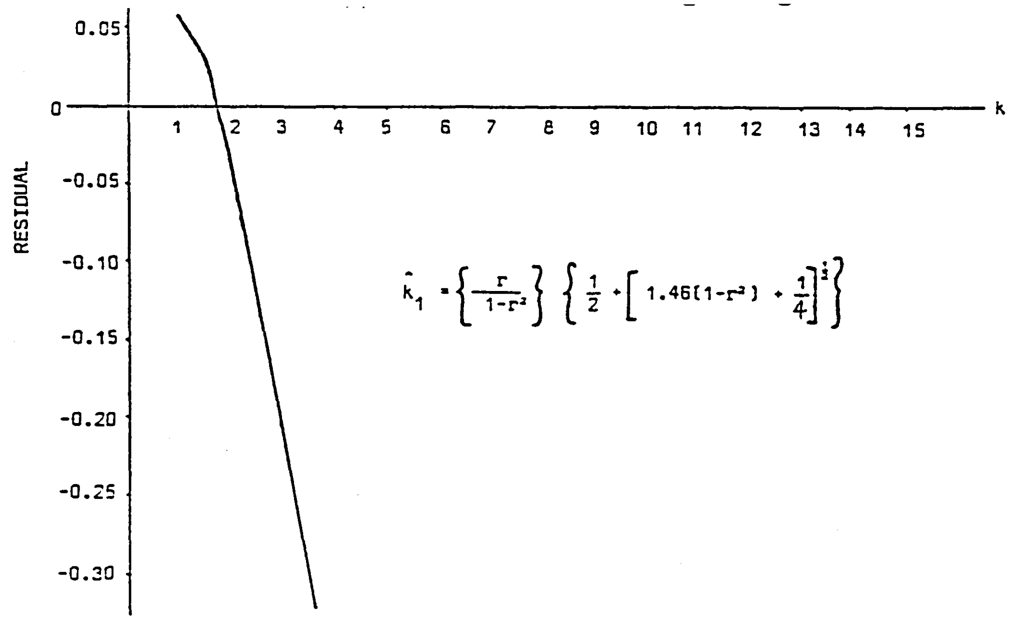


Figure 3.3(a)

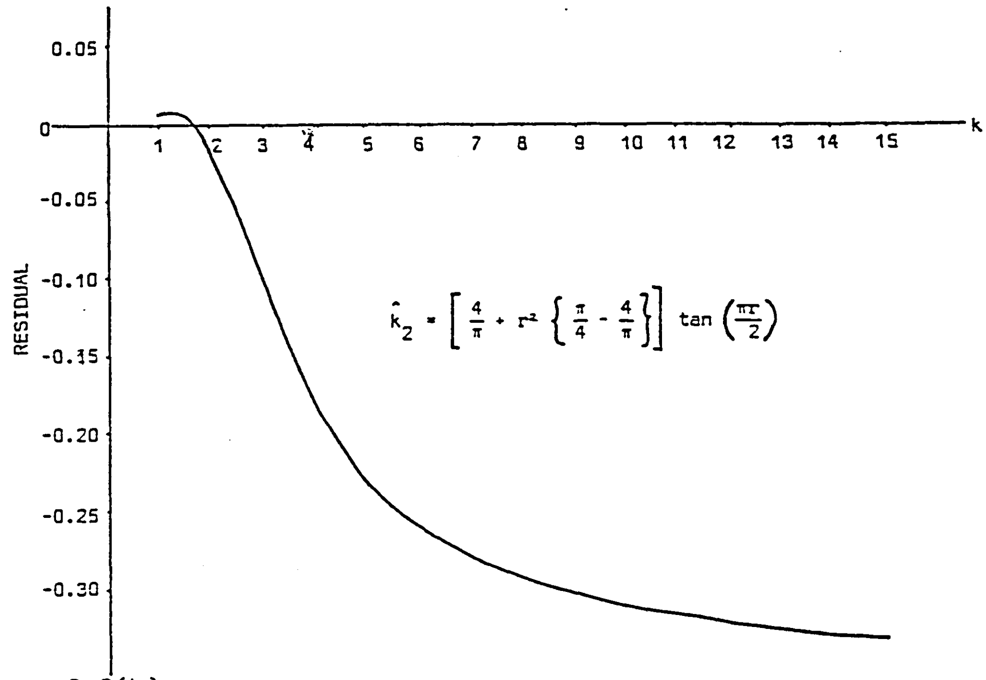


Figure 3.3(b)

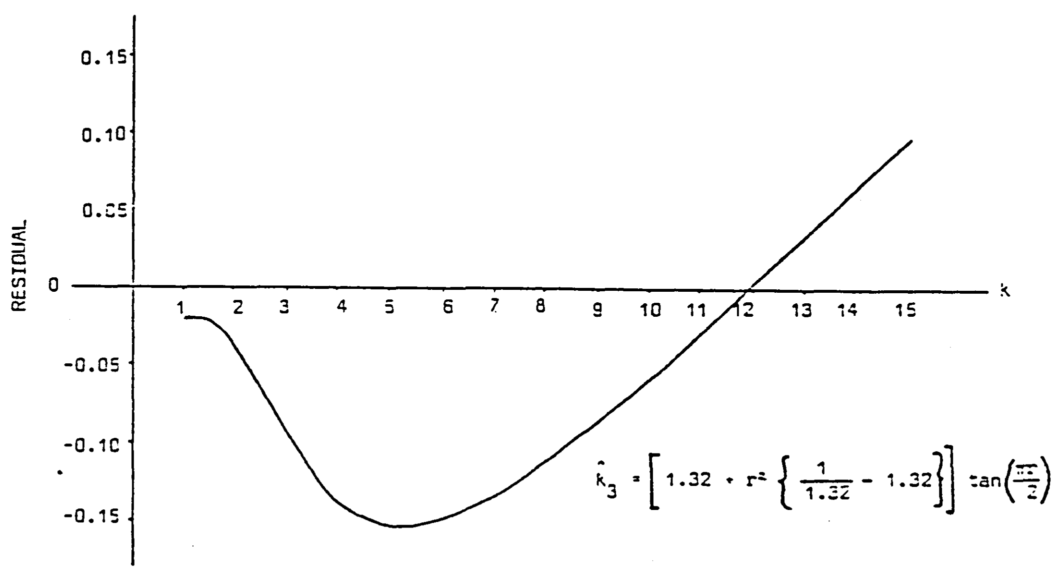


Figure 3.3(c)

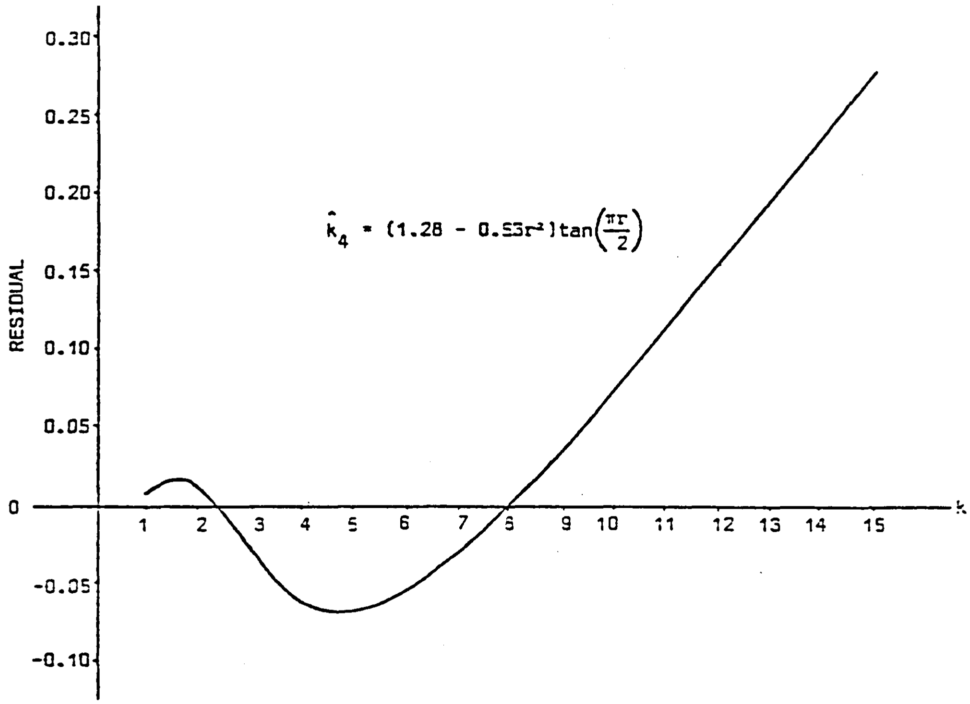


Figure 3.3(d)

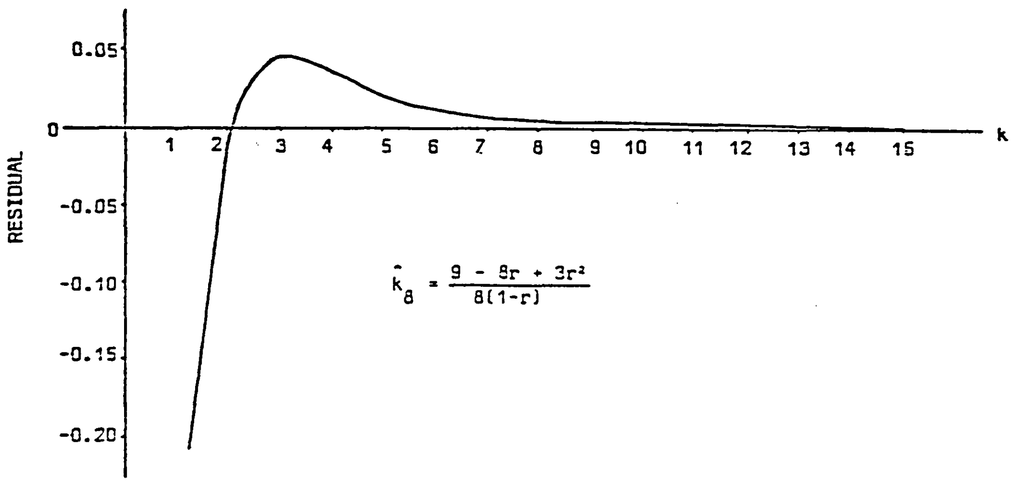


Figure 3.3(e)

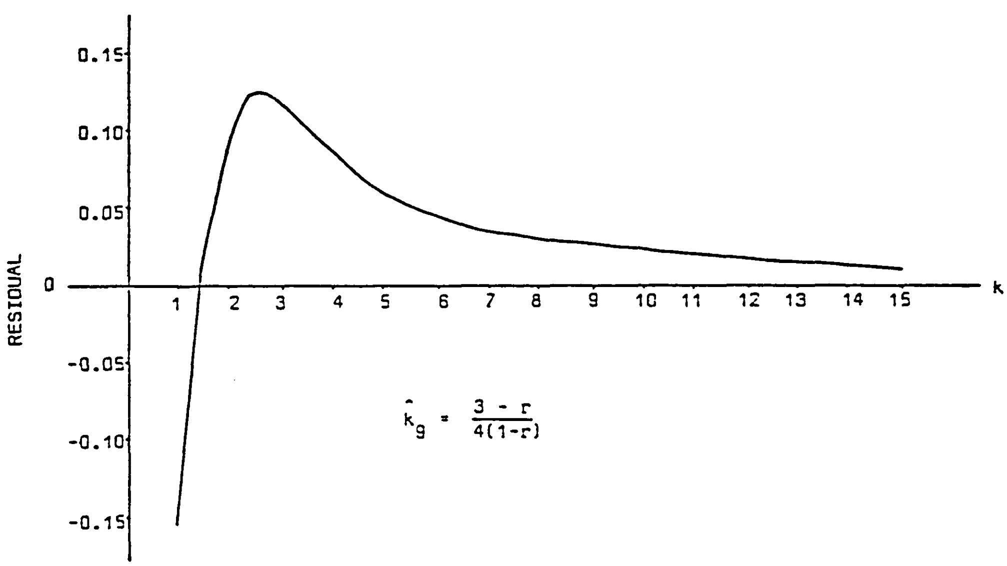


Figure 3.3(f)

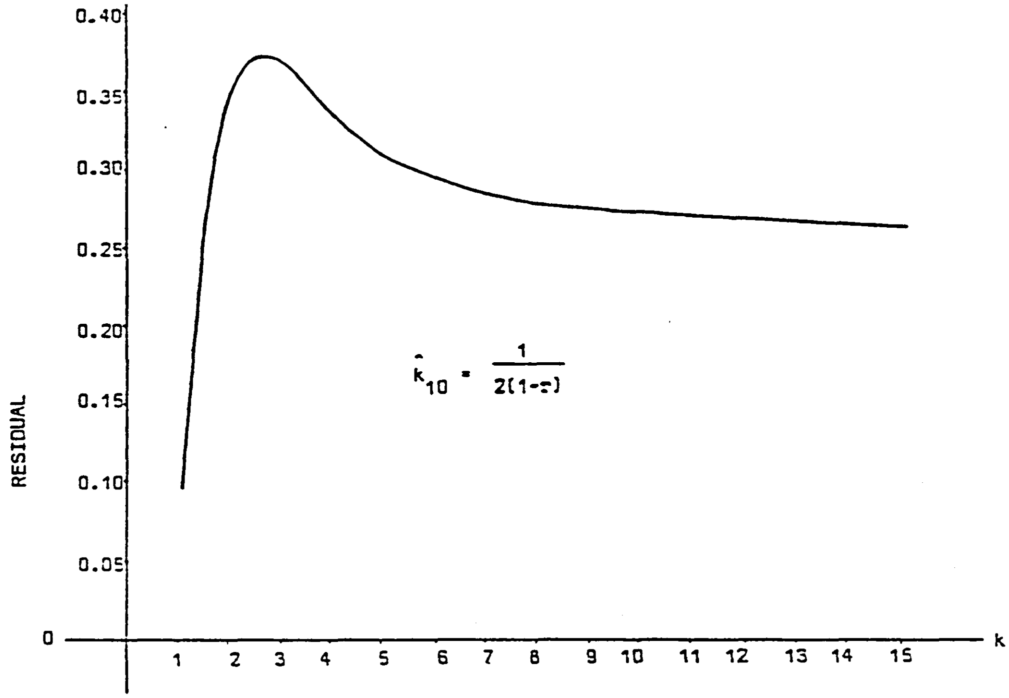


Figure 3.3(g)

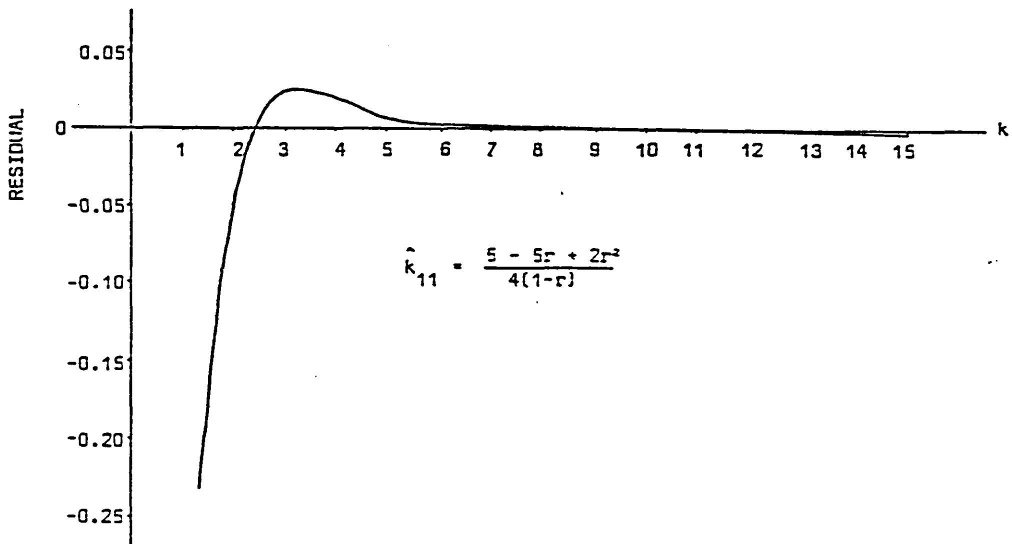


Figure 3.3(h)

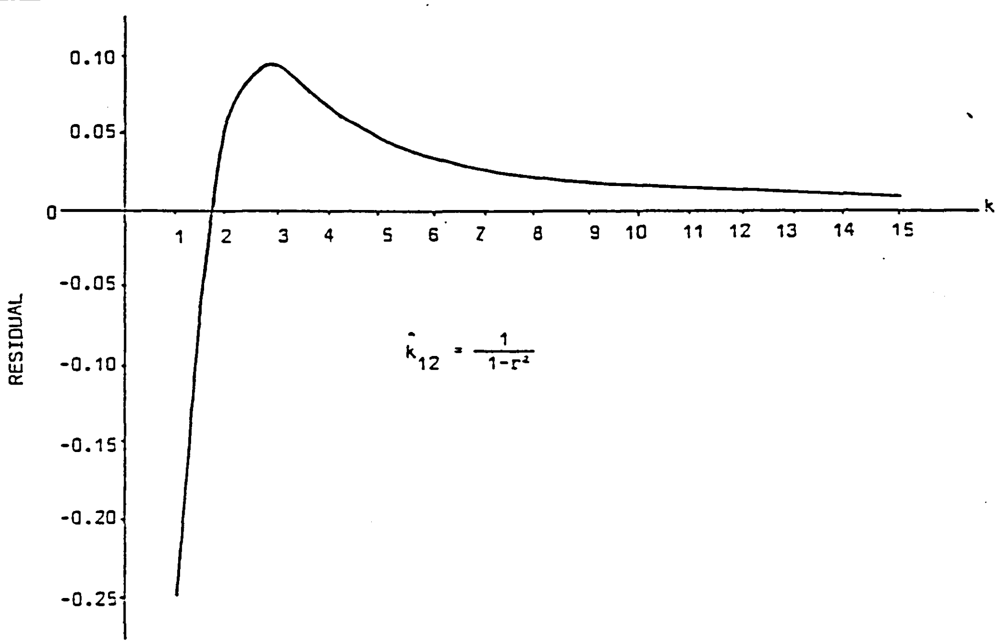


Figure 3.3(i)

the approximations for large k against k.

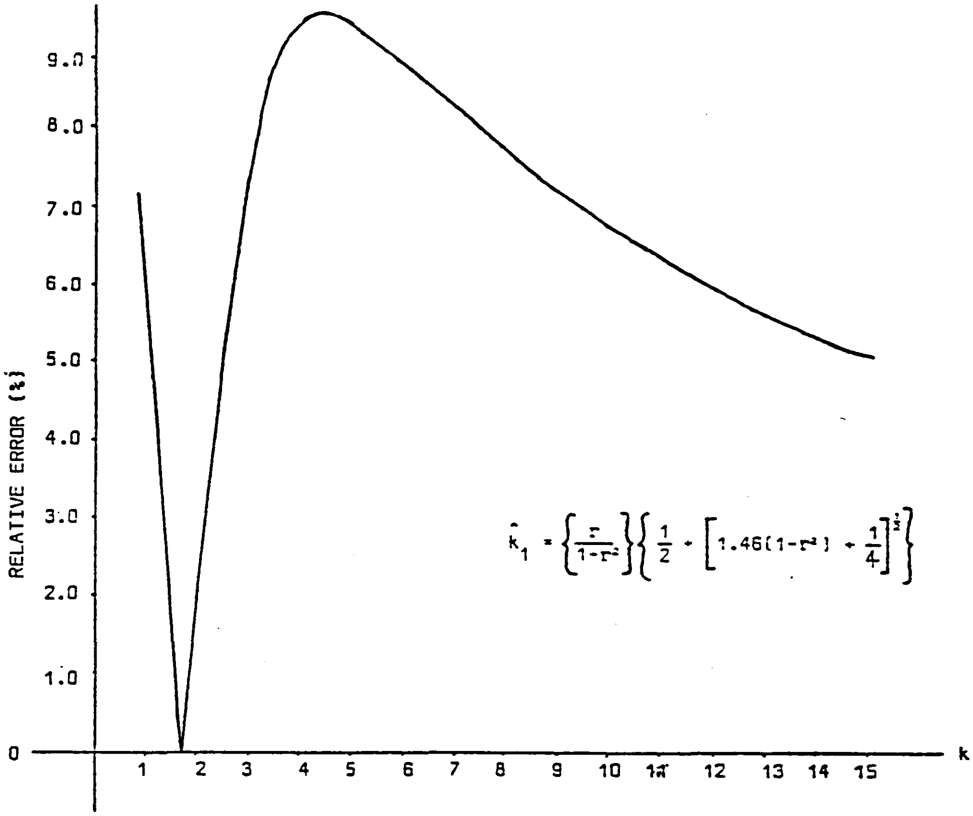


Figure 3.4(a)

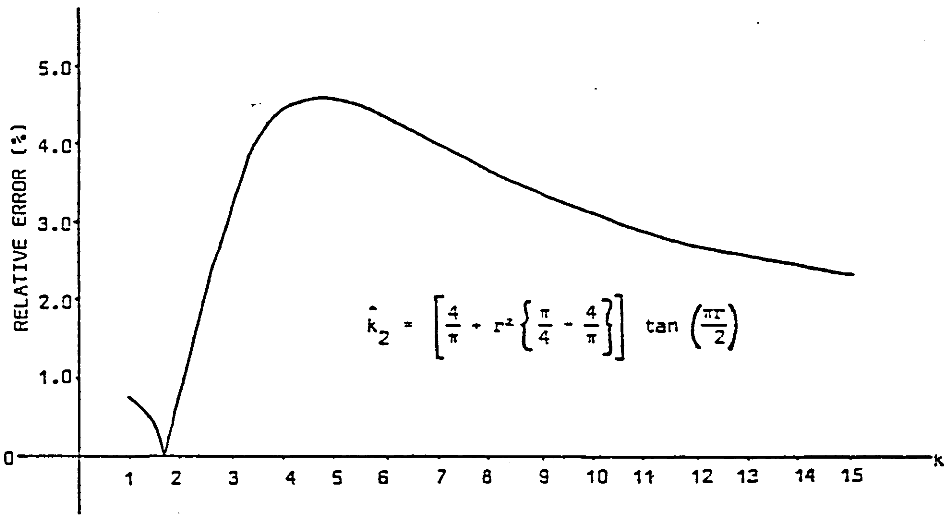


Figure 3.4(b)

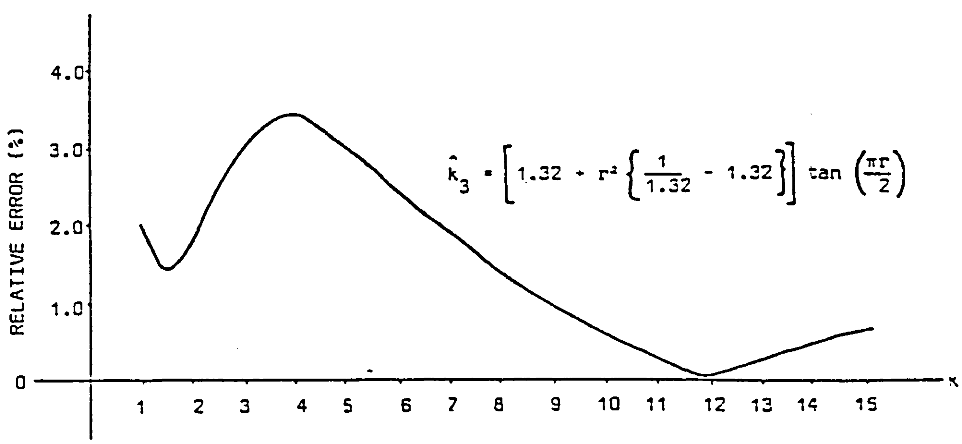


Figure 3.4(c)

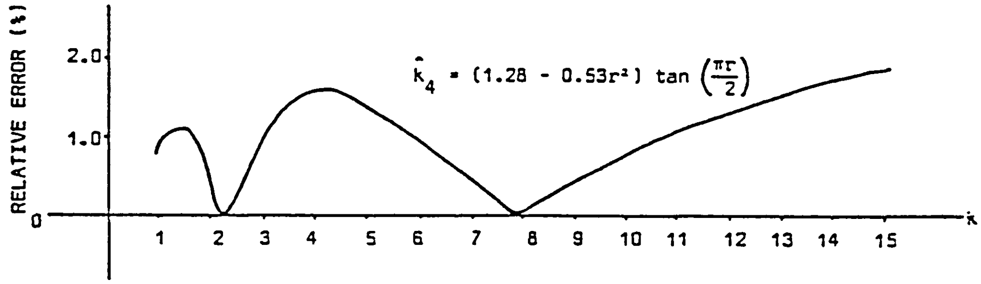


Figure 3.4(d)

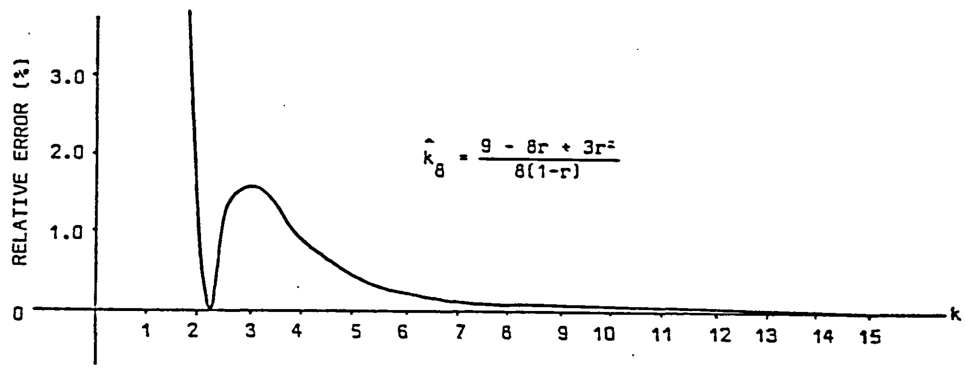
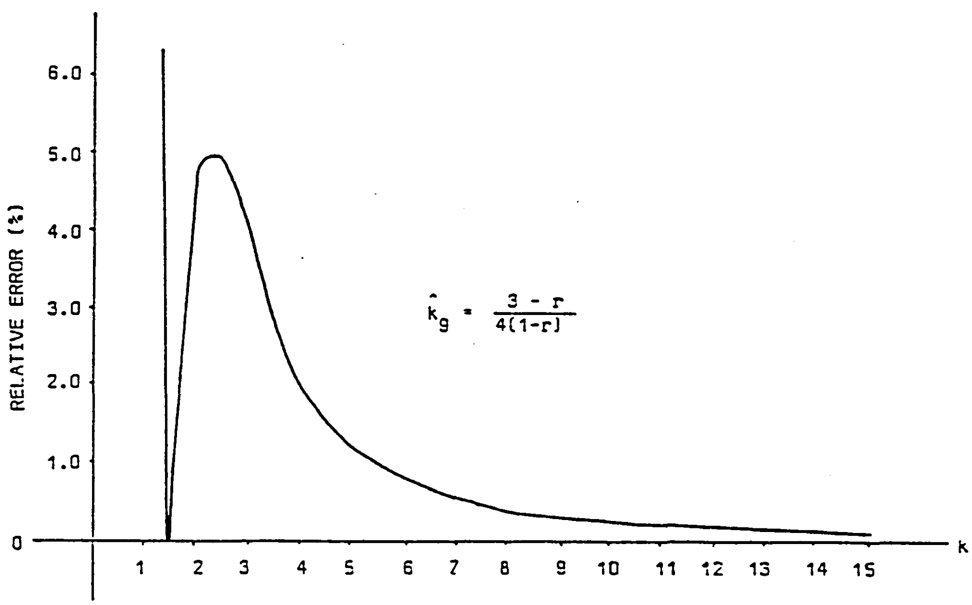


Figure 3.4(e)



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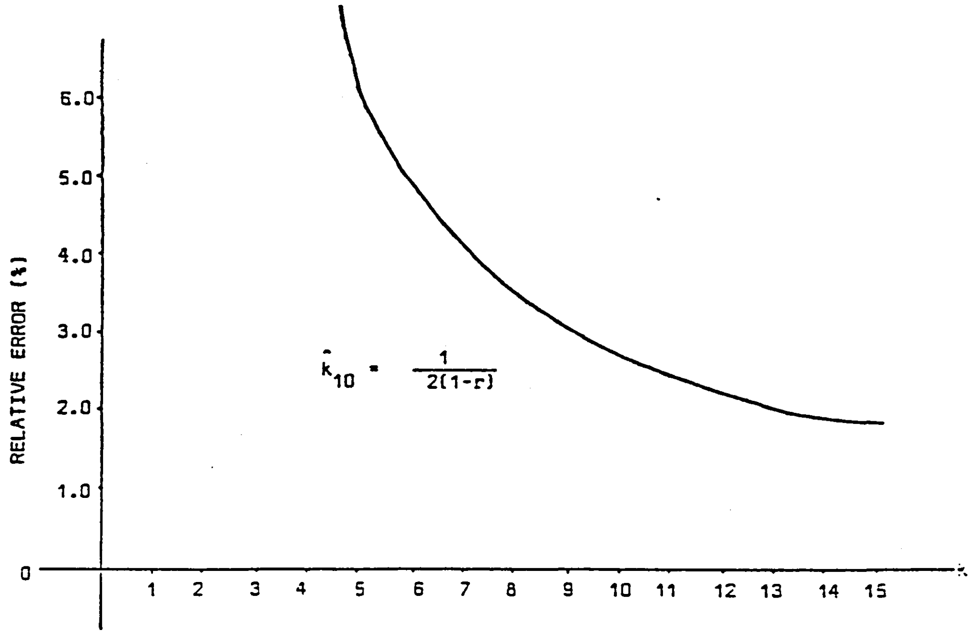


Figure 3.4(g)

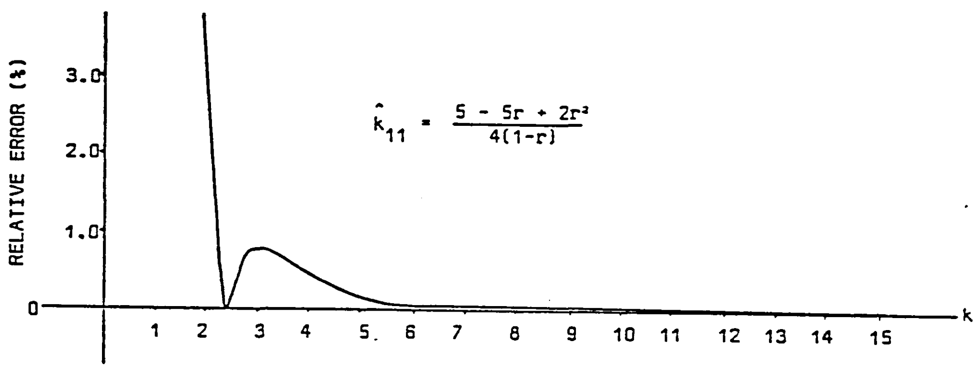


Figure 3.4(h)

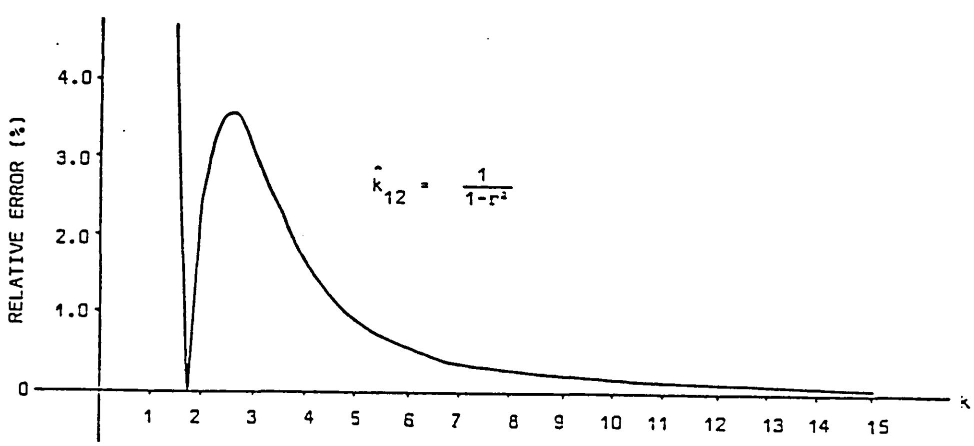


Figure 3.4(i)

3.4 The Best Approximation for the von Mises Concentration Statistic, $\hat{k}$, in the Range $1 < k < 2.5$

From examining the results and graphs for the approximations for large and small k, all the power series functions are very poor estimators in the range $1 < k < 2.5$.

From the global approximations, Figures 3.3(a) to 3.3(d) and 3.4(a) to 3.4(d), $\hat{k}_1$ is a poor approximation and may be removed. Taking a closer examination of $\hat{k}_2$, $\hat{k}_3$ and $\hat{k}_4$ in the range $1 < k < 2.5$, the residuals for these are given in Figures 3.5(a) to (c) respectively. Here we can see that all three are good approximations, however, $\hat{k}_4$ is the best with maximum residual of 0.0168 at k=1.6. The maximum percentage relative errors for $\hat{k}_2$, $\hat{k}_3$ and $\hat{k}_4$ in this range are, 2.125%, 2.446% and 1.12% respectively.

approximations to k in the range $1 \leq k \leq 2.5$

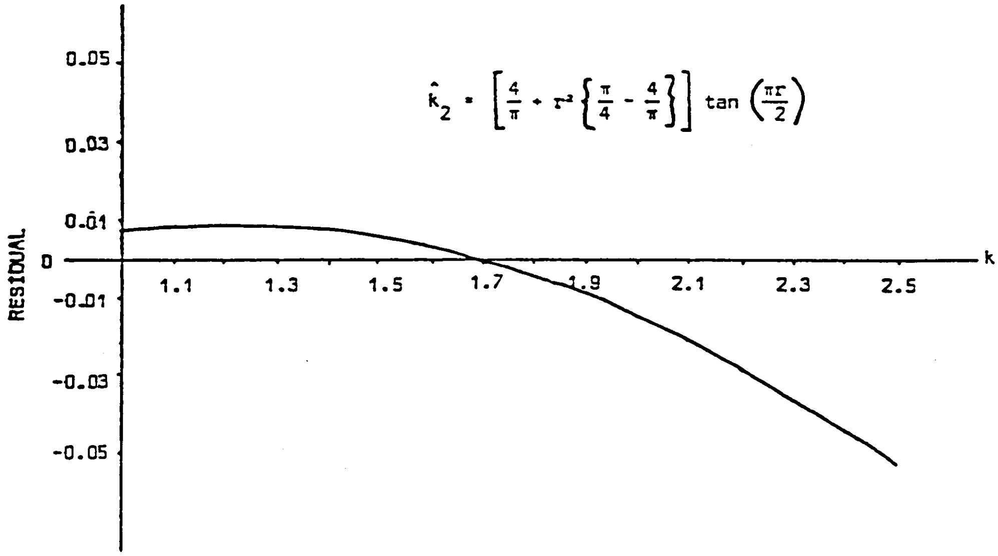


Figure 3.5(a)

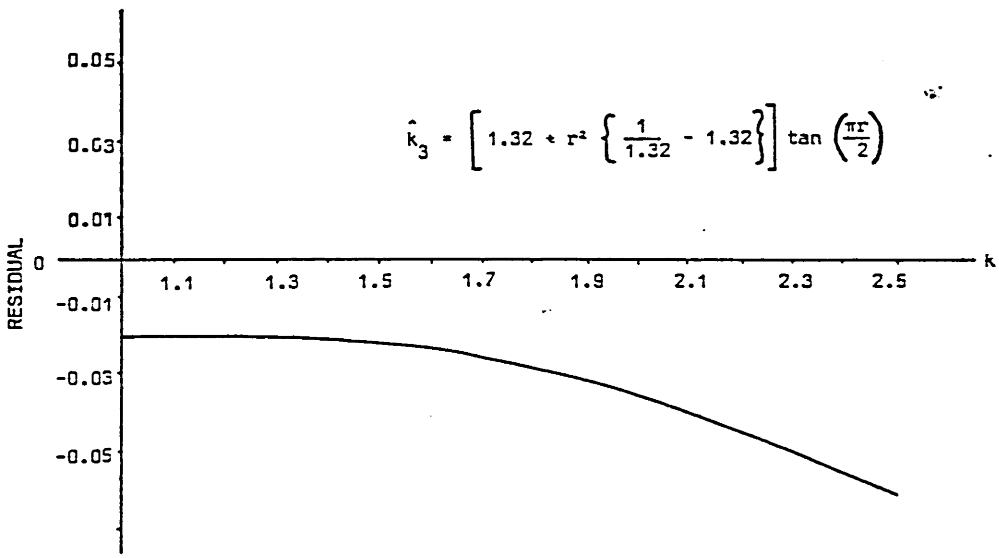


Figure 3.5(b)

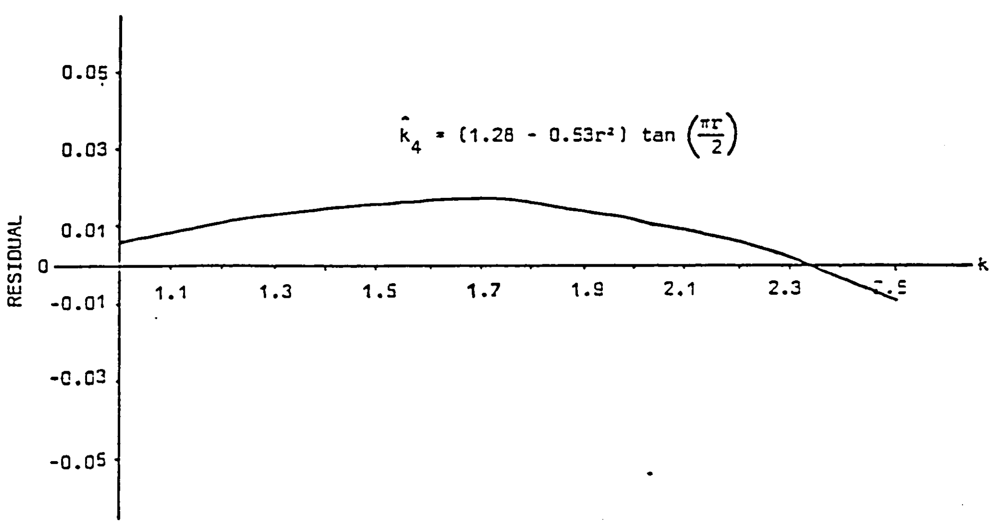


Figure 3.5(c)

3.5 Summary of Approximations

For the twelve approximations examined, Table 3.5.1 gives the best three approximations of k with their respective ranges. Also shown are the best 'simple' approximations of k.

TABLE 3.5.1

<table>
<tr><th>Range of k</th><th>Best Function Approximation</th><th>Range of k</th><th>Best 'Simple' Function Approximation</th></tr>
<tr><td rowspan="2">$0 < k \leqslant 1.25$
or
$0 < r \leqslant 0.528$</td><td rowspan="2">$r[2+r^2 + \frac{5r^4}{6}]$</td><td>$0 < k < 0.2$
or
$0 < r \leqslant 0.1$</td><td>$2r$</td></tr>
<tr><td rowspan="2">$0.2 < k < 1.45$
or
$0.1 < r < 0.584$</td><td rowspan="2">$r[2+r^2]$</td></tr>
<tr><td rowspan="2">$1.25 < k \leqslant 2.45$
or
$0.528 < r \leqslant 0.759$</td><td rowspan="2">$[1.28-0.53r^2]\tan\left[\frac{\pi r}{2}\right]$</td></tr>
<tr><td rowspan="2">$k \geqslant 1.45$
or
$r \geqslant 0.584$</td><td rowspan="2">$\frac{1}{1-r^2}$</td></tr>
<tr><td>$k > 2.45$
or
$r > 0.759$</td><td>$\frac{5-5r+2r^2}{4(1-r)}$</td></tr>
</table>

Figures 3.6(a) and (b) plot the absolute residuals for the best and best 'simple' approximations in their respective ranges as continuous functions.

Figure 3.6(a). Plot of the absolute residuals for the best approximations to concentration statistic, k

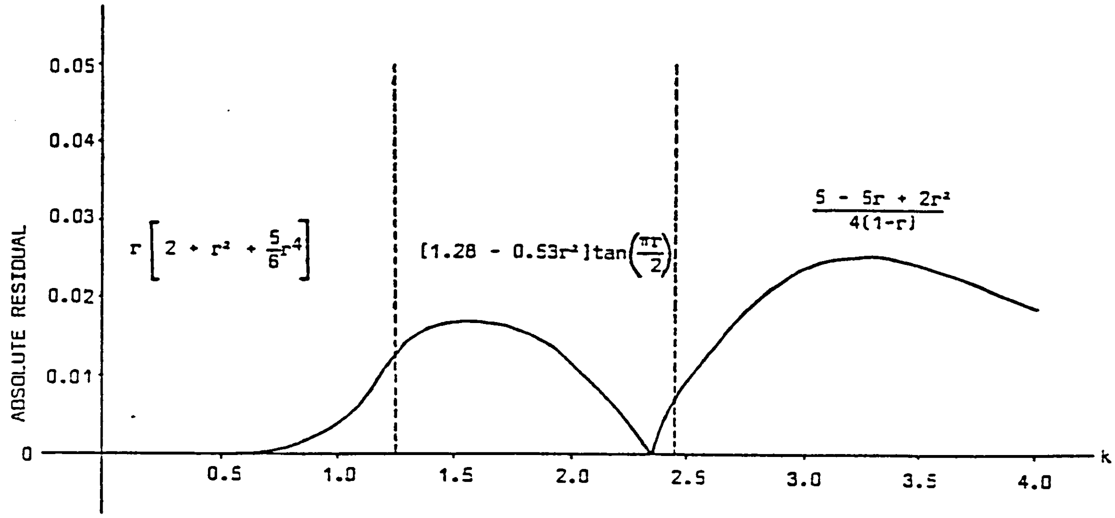
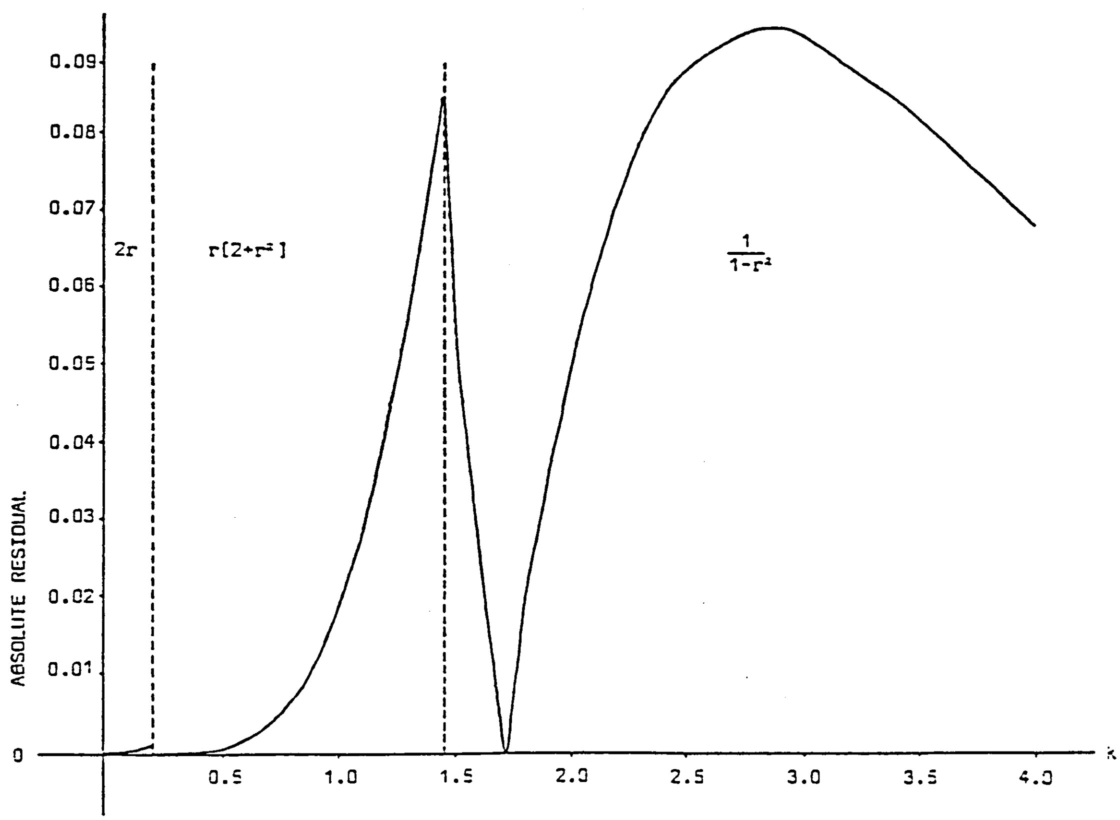


Figure 3.6(b). Plot of the absolute residuals for the best 'simple' approximation to concentration statistic, k



Naturally we obtain better and better approximations using further terms in the power series (3.3.4) and (3.3.9), however we wish to find and use fairly simple approximations to (3.2.2) in order that tables or large computing equipment are not required.

3.6 Expansion of $[I_0(A)/I_0(B)]^N$

When calculating the likelihood ratio test for large or small values of k an expansion and approximation of (3.6.1) is required.

$$N \log\left\{\frac{I_0(A)}{I_0(B)}\right\} \tag{3.6.1}$$

If A and B are both small we may use the standard series expansion of Bessel functions quoted by Bickley (1957) to obtain

$$\frac{I_0(A)}{I_0(B)} = \left[1 + \frac{A^2}{4} + \frac{A^4}{64} + \dots\dots\right]\left[1 + \frac{B^2}{4} + \frac{B^4}{64} + \dots\dots\right]^{-1}$$

$$\approx 1 + \left[\frac{1}{4}\right](A^2 - B^2) + \left[\frac{1}{64}\right](A^4 - 4A^2B^2 + 3B^4) \tag{3.6.2}$$

For (3.6.1) the power series for log(1+x) is used to produce

$$N \log\left\{\frac{I_0(A)}{I_0(B)}\right\} \approx \left[\frac{N}{4}\right](A^2 - B^2) - \left[\frac{N}{64}\right](A^4 - B^4) \tag{3.6.3}$$

For very small values of A and B the second term may be neglected to give

$$N \log\left\{\frac{I_0(A)}{I_0(B)}\right\} \approx \left[\frac{N}{4}\right](A^2 - B^2) \tag{3.6.4}$$

For large values of A and B the Bessel functions are expanded using the asymptotic expansion (3.3.1) to obtain

$$\frac{I_0(A)}{I_0(B)} \approx \left[\frac{B}{A}\right]^{\frac{1}{2}} \exp[A - B] \left\{1 + \frac{1}{8}\left[\frac{1}{A} - \frac{1}{B}\right] + \frac{1}{128}\left[\frac{9}{A^2} - \frac{2}{AB} - \frac{7}{B^2}\right]\right\} \qquad (3.6.5)$$

For (3.6.1) we again expand the logorithm as a power series.

$$N \log\left\{\frac{I_0(A)}{I_0(B)}\right\} \approx \left[\frac{N}{2}\right] \log\left[\frac{B}{A}\right] + N(A - B)\left[1 - \frac{1}{8AB} - \frac{(A + B)}{16A^2B^2}\right] \qquad (3.6.7)$$

For very large A and B we may neglect the higher powers of 1/A and 1/B to give

$$N \log\left\{\frac{I_0(A)}{I_0(B)}\right\} \approx \left[\frac{N}{2}\right] \log\left[\frac{B}{A}\right] + N(A - B) \qquad (3.6.7)$$

CHAPTER 4

THE DEVELOPMENT OF CIRCULAR ANALYSIS OF VARIANCE TECHNIQUES

4.1 Introduction

The analysis of data of a circular or spherical nature began when Lord Rayleigh (1880) developed a one-sample test for uniformly distributed random vectors using the resultant length. Investigation into circular distributions other than the uniform distribution began in the early part of this century. The most important of which, and the assumed circular distribution for much of this thesis, was the von Mises distribution. True development of significance tests, however, did not appear until Fisher (1953), whilst investigating the remanent magnetism of a sample of rock specimens, considered the spherical analogue of the von Mises distribution where observations are regarded as points on a sphere. Fisher derived the maximum likelihood estimates of the concentration parameter and the mean direction and provided the basic distribution theory in order to test a prescribed mean direction when k is unknown. Watson (1956) gave a significance table for the test of $k = 0$ i.e. uniformity, and approximate tests for the equality of concentration parameters and mean directions. As discussed in Chapter 2.3 Greenwood and Durand (1955) utilised Fishers work to produce a similar distribution theory for the circular case. In 1956 Watson and Williams derived tests for the direction and homogeneity in both the two- and three-dimensional cases. Their exact test for the two-dimensional case is summarised in the following section.

4.2 Exact and Approximate Tests of Significance from Watson and Williams

4.2.1 An Exact Test

Stephens (1972) produced an exact two-sample test for the null hypothesis that two samples, size N_1 and N_2, have identical modal vectors, assuming they have the same unknown value of k. For each sample the resultant lengths are found and the test statistic Z calculated

$$Z = \frac{R_1 + R_2}{N} \tag{4.2.1}$$

The resultant R of R_1 and R_2 is also found and $W = R/N$ calculated.

The test consists of finding a critical value z satisfying $\Pr(Z > z/W) = \alpha$, for appropriate significance level α. Tables for the critical value z are presented by Stephens (1972) and the null hypothesis is rejected at level α if $Z = z$. The exact test is a conditional test based on the joint distribution of R_1, R_2 given R i.e.

$$f(R_1, R_2 | R) = \frac{2R\, p_{u_1}(R_1) p_{u_2}(R_2)}{\pi p_u(R)\{[(R_1 + R_2)^2 - R^2][R^2 - (R_1 - R_2)^2]\}^{\frac{1}{2}}} \tag{4.2.2}$$

where

$$0 \leqslant R_1 \leqslant n_1 \qquad 0 \leqslant R_2 \leqslant n_2 \qquad |R_1 - R_2| < R < R_1 + R_2$$

and $p_u(R)$ is given by equation (2.3.9). Equation (4.2.2) was derived by Watson and Williams (1956) for the circle following the derivation for the sphere by Fisher (1953).

4.2.2 Approximate Tests

If θ is an observation from the von Mises distribution with mode at zero then for large k we have shown, (2.2.4), that $\theta\sqrt{k}$ is approximately N(0,1) and hence $k\theta^2$

to be chi-squared distribution with 1 degree of freedom. Using this result we may approximate for θ^2 to obtain the result that $2k(1-\cos\theta)$ is chi-squared distributed with 1 degree of freedom.

Watson and Williams used this result and the additive properties of χ^2 distributions to produce the approximations

$$\left.\begin{array}{lll} 2k(1 - \cos\theta) & \approx & \chi_1^2 \\ 2k(N - X) & \approx & \chi_N^2 \\ 2k(N - R) & \approx & \chi_{N-1}^2 \end{array}\right\} \qquad (4.2.3)$$

for single sample tests.

In the two sample case, if $\bar{\theta}_1$ (the mean direction for sample 1) equals $\bar{\theta}_2$ (the mean direction for sample 2) then

$$R_1 + R_2 = R$$

However, if $\bar{\theta}_1 \neq \bar{\theta}_2$

then $\quad R_1 + R_2 > R$

Therefore the greater the difference between $\bar{\theta}_1$ and $\bar{\theta}_2$ the greater the value of $R_1 + R_2 - R$. Via equations (4.2.3), Watson and Williams showed

$$\left.\begin{array}{lll} 2k(R_1 + R_2 - R) & \approx & \chi_1^2 \\ 2k(N - (R_1 + R_2)) & \approx & \chi_{N-2}^2 \end{array}\right\} \qquad (4.2.4)$$

where $\quad N = \sum_{j=1}^{q} n_j$

Equations (4.2.4) are independently distributed. (Further proof of (4.2.4) and (4.2.3) can be seen in Mardia (1972 p.114)).

Watson and Williams then stated that to test the equality of mean vectors for several samples (for large k), generalising Watson (1956), we should use

$$\frac{(N - q)(\sum_{j=1}^{q} R_j - R)}{(q - 1)(N - \sum_{j=1}^{q} R_j)} = F_{q-1,N-q} \tag{4.2.5}$$

where q is the number of samples.

4.3 Hypothesis Testing Concerning the Mean Direction

In 1962 two papers by M A Stephens produced exact and approximate tests for the null hypothesis, concerning single sample tests, that the mean vector is a given vector when k is unknown. For the exact test, given in section 4.2.1, Stephens produced nomograms for differing significance levels to find R_0 given N and X, where $Pr(P > R_0 | X) = \alpha$. X being the component of R on $\bar{\theta}$, when μ_0 is known (Figure 2.2.1).

Stephens' three approximate tests were also for the above null hypothesis, for different ranges of N and X. For large concentration parameter the approximate test was given by

$$(N - 1)\frac{(R_0 - X)}{(N - R_0)} = F_{1,N-1}(\alpha) \tag{4.3.1}$$

calculating R_0 from (4.3.1). If $R > R_0$, reject the null hypothesis.

Equation (4.3.1) was first suggested by Watson and Williams using the equations of (4.2.3). Using these equations the chi-squared approximations may also be shown as;

$$2k(N - X) = 2k(R - X) + 2k(N - R) \tag{4.3.2}$$

For large k, this obeys the χ^2 decomposition property parallel to

$$\frac{1}{\sigma^2}\sum_{i=1}^{N}(x_i - \mu)^2 = \frac{1}{\sigma^2}(\bar{x} - \mu)^2 + \frac{1}{\sigma^2}\sum_{i=1}^{N}(x_i - \bar{x})^2 \tag{4.3.3}$$

where $x_1,\ldots,x_N$ is a random sample from $N(\mu,\sigma)$ for linear statistics.

Equation (4.3.1) is therefore similar to the F–statistic for linear statistics, which is the ratio square–root of the first and second terms on the right–hand side of (4.3.3).

In 1969 Stephens stated that in practice the asympototic results of (4.2.3) are not adequate for moderately large values of k. Illustrating this via Pearson curve approximations and Monte Carlo studies, an improvement in (4.2.3) was found by examining the expansion of $A(k)$ for large k (equation (3.2.2)). This suggested that the chi–squared form would be improved by replacing k by γ in (4.2.3), where

$$\frac{1}{\gamma} = \frac{1}{k} + \frac{3}{8k^2} \tag{4.3.4}$$

and that this should be used for tests when $k > 2$.

In 1974 Upton considered further improvements of (4.2.3) by investigating the distributions of θ, X and R. Taking the expectation of these distributions and approximating the Bessel functions involved, Upton derived β, to replace k in (4.2.3).

$$\beta = k\left[1 - \frac{1}{4k} - \frac{3}{16k^2}\right] \tag{4.3.5}$$

Upton concluded that (1) both γ and β approximations improve as k increases (2) both γ and β are considerable improvements over the original, and (3) there is little to choose between the β and γ approximations

Assuming equal concentration parameters k_1 and k_2 for the two sample case, we are now interested in testing

$$H_0 : \mu_{01} = \mu_{02} = \mu_0 \tag{4.3.6}$$

against

$$H_1 : \mu_{01} \neq \mu_{02} \tag{4.3.7}$$

where μ_0 and k are unknown.

Using Stephens approximation improvement γ, for large k, (4.2.5) from Watson and Williams becomes

$$F_{1,N-2} = \left[1 + \frac{3}{8k}\right] (N-2) \left[\frac{(R_1 + R_2 - R)}{(N - R_1 - R_2)}\right] \tag{4.3.8}$$

where for unknown k, k can be replaced by its maximum likelihood estimate,

$$\hat{k} = A^{-1}\left[\frac{R}{N}\right]$$

given by (3.3.11) or (3.3.10).

For value of $\hat{k} > 10$ (R/N > 0.95) the improvement factor γ is negligible.

Using the likelihood ratio test procedure, discussed in Chapter 3.1, to test H_0 and H_1 defined by (4.3.6) and (4.3.7) we may obtain a test statistic for small k, assumed equal, given by

$$\log \lambda = N \log\left[\frac{I_0(k_1)}{I_0(k_0)}\right] + k_0 \sum_{i=1}^{p} \sum_{j=1}^{2} \cos(\theta_{ij} - \bar{\theta})$$

$$- k_1 \sum_{i=1}^{p} \sum_{j=1}^{2} \cos(\theta_{ij} - \bar{\theta}_j) \tag{4.3.9}$$

From (1.4.11)

$$\sum_{i=1}^{p} \sum_{j=1}^{2} \cos(\theta_{ij} - \bar{\theta}) = R$$

$$\sum_{i=1}^{p} \sum_{j=1}^{2} \cos(\theta_{ij} - \bar{\theta}_j) = R_1 + R_2$$

Using (3.2.11) for small k

$$\left.\begin{aligned} k_0 &= \frac{2R}{N} \\ k_1 &= \frac{2(R_1 + R_2)}{N} \end{aligned}\right\} \tag{4.3.10}$$

where

$$N = N_1 + N_2$$

Using approximation (3.6.4) we obtain the test statistic

$$-2\log \lambda = \frac{2}{N}\left\{(R_1 + R_2)^2 - R^2\right\} = \chi_1^2 \tag{4.3.11}$$

In the same manner the likelihood ratio test of the equality of the mean directions of two samples having unknown concentration parameters, assumed to be equal and large, may be obtained from (4.3.9).

With k assumed large we may use the approximation (3.3.7) to give

$$\left.\begin{aligned} k_0 &\approx \frac{N}{2(N - R)} \\ k_1 &\approx \frac{N}{2(N - R_1 - R_2)} \end{aligned}\right\} \tag{4.3.12}$$

Using the approximation (3.6.7) for large k we obtain our test statistic

$$N\log\left\{\frac{N - R}{N - R_1 - R_2}\right\} \approx \chi_1^2 \qquad (4.3.13)$$

Mardia (1972) shows this to be a monotonic function of the F–statistic given by (4.3.8).

For both the single and two sample cases Upton (1970, 74), using likelihood ratio techniques, produced significance tests concerning the mean direction and the concentration parameter for all permutations of k known or unknown, k large or small, and mean direction known or unknown. Upton also improved these test statistics using their expectations and associated degrees of freedom, as discussed in Chapter 3.1

Many single and two sample tests have not been fully reviewed and reproduced here as this thesis is concerned with the further development of analysis of variance techniques and only those having a bearing on such development have been introduced. An excellent review of single and two sample testing can be seen in Mardia (1972, p 132–).

4.4 Multi–sample Tests Concerning the Mean Direction

Let θ_{ij} (i=1,2.....p,j=1,2,.....,q) be q independent random samples of sizes N_j from $M(\mu_{0j},k_j)$. Let R_j be the length of the resultant of the jth sample, and R be the length of the resultant of the whole or combined sample.

We wish to test

$$H_0 : \mu_{0,1} = \mu_{0,2} = \ldots\ldots = \mu_{0,q} \qquad (4.4.1)$$

against the alternative that at least one of the equalities does not hold. We assume that $k_1 = \ldots k_q = k$, where k is unknown.

4.4.1 For Small Concentration Parameter, k

For values of k in the range 0 to 1 the likelihood ratio test for q samples extended from (4.3.11) gives

$$-2\log\lambda = \frac{2}{N}\left(\left(\sum_{j=1}^{q} R_j\right)^2 - R^2\right) = \chi^2_{q-1} \qquad (4.4.2)$$

Since this test uses the approximation (3.2.11) for small k, which was shown in Chapter 3 to be unreliable as $k \to 1$, the test may be improved by using the expectation approximations from (2.5.2) and (2.5.4)

$$E(R_j) = N_j\rho + \frac{1}{2k} \qquad k > 0$$

$$E(R_j^2) = N_j + N_j(N_j - 1)\rho^2$$

in (4.4.2) to produce the test statistic

$$\frac{2}{N}\,\delta\left(\left(\sum_{j=1}^{q} R_j\right)^2 - R^2\right) \qquad (4.4.3)$$

where

$$\delta^{-1} = 1 - \frac{k^2}{8} + \frac{q}{2Nk^2} \qquad (4.4.4)$$

4.4.2 For Large Concentration Parameter, k

Using Stephens improvement (4.3.4) and extending Watson and Williams test statistic of (4.2.5), the new test statistic, under the null hypothesis (4.4.1), becomes

$$\left[1 + \frac{3}{8\hat{k}}\right]\left\{\frac{(N - q)\left(\sum_{j=1}^{q} R_j - R\right)}{(q - 1)\left(N - \sum_{j=1}^{q} R_j\right)}\right\} = F_{q-1,N-q} \qquad (4.4.5)$$

where $\hat{k}$ is the m.l.e. of k given in (3.2.2).

Mardia (1972) states that from Monte Carlo trials this approximation is adequate for $k \geqslant 1$, i.e R/N $\geqslant$ 0.45 Stephens (1969, 72), however, shows that the test is adequate for $k \geqslant 2$, i.e. R/N $\geqslant$ 0.7, and requires a further improvement to be satisfactory for $k \geqslant 1$. Stephens (1972) produced an approximate multi-sample test based on the exact test given in section 4.2.1, where

$$Z = \frac{R_1 + R_2 + \ldots\ldots\ldots + R_q}{N} \quad \text{and} \quad W = \frac{R}{N} \tag{4.4.6}$$

With test statistic;

$$z = \frac{W + f}{1 + f} \tag{4.4.7}$$

where

$$f = \frac{g\ D(q - 1)}{(N - q)}$$

Let g be the upper percentage point, at level α, of the F distribution with q–1 and N–q degrees of freedom, and let D be a parameter taking the following values, for W between 0.45 and 1.0.

W :	0.45	0.50	0.55	0.60	0.65	0.70	0.80	0.90	1.00
D :	0.92	0.87	0.84	0.83	0.82	0.84	0.88	0.93	1.00

If Z > z, reject H_0 at significance level α

This test is equivalent to (4.4.5) for $k \geqslant 2$, but as (4.4.5) deviates from the F–distribution below k=2 the factor D is introduced. For values of $k \geqslant 2$ D is the reciprocal of the factor 1 + (3/8k).

Upton (1976) produced an approximate test for q samples, known as the G–test, related to the joint distribution of Z and W, where Z and W are given by (4.4.6).

The G–test is derived from the method of maximum likelihood which gave (4.3.13) for the two–sample case. Upton states the test statistic

$$G = \frac{2N^2 \log\left\{\dfrac{1 - W}{1 - Z}\right\}}{N(1 + W) + q} = \chi^2_{q-1} \tag{4.4.8}$$

suitable when all $N_j \geqslant 10$ and $R/N \geqslant 0.6$.

In exactly the same procedure as the two–sample case, $\hat{k}_0$ and $\hat{k}_1$ are produced from (3.3.7) to give

$$\left.\begin{aligned} \hat{k}_0 &= \frac{1}{2(1 - W)} \\ \hat{k}_1 &= \frac{1}{2(1 - Z)} \end{aligned}\right\} \tag{4.4.9}$$

Using the approximation (3.6.7) for large k and the log likelihood ratio

$$N\left[\log\left[\frac{I_0(\hat{k}_1)}{I_0(\hat{k}_0)}\right] + \hat{k}_0 W - \hat{k}_1 Z\right]$$

we obtain the test statistic

$$-2 \log\lambda = N \log\left\{\frac{(1 - W)}{(1 - Z)}\right\} \tag{4.4.10}$$

This may be improved by equating the test statistic to its associated chi–squared expectation. On expanding the logarithm as a power series we may neglect terms beyond the first two since W and Z will be smaller than 1, to produce

$$N\left\{- \frac{W}{2} - W^2 + Z + \frac{Z^2}{2}\right\}$$

using the expectations $E(R)$ and $E(R^2)$ given by equations (2.5.4) and (2.5.2).

$$E(G) = \frac{(q - 1)}{2k}\left[1 - A(k) + \frac{q}{4kN}\right]$$

On equating with (q−1) and simplifying we obtain the correction

$$\frac{2N}{N + R + q}$$

to produce the test statistic (4.4.8).

Throughout Uptons paper, conditions, ranges and tables for the statistic W/N are given, however, this is a notation error and should be read as R/N or simply W.

The above multi-sample tests will be investigated further in Chapter 9 when suitable comparisons to an alternative test will be given.

4.5 Multi-Sample Tests for the Equality of Concentration Parameters, k_j

In this section three tests for the homogeneity of concentration parameters for differing values of R_j/N_j are given. The construction of tests (4.5.2), (4.5.4) and (4.5.5) can be seen in Mardia (1972, p 165−). The composite hypothesis under consideration is

$$H_0 : k_1 = \ldots\ldots = k_q = k \tag{4.5.1}$$

where $\mu_1, \ldots\ldots\mu_q$ and k are not specified.

4.5.1 $R_j/N_j < 0.45$

Test statistic

$$U_1 = \sum_{j=1}^{q} w_j g_1^2(\tilde{R}_j) - \frac{\left(\sum_{j=1}^{q} w_j g_1(\tilde{R}_j) \right)^2}{\sum_{j=1}^{q} w_j} = \chi^2_{q-1} \tag{4.5.2}$$

where

$$\tilde{R}_j = \frac{2R_j}{N_j} \qquad g_1(\tilde{R}_j) = \sin^{-1}(a\tilde{R}_j) \qquad a = \left[\frac{3}{8}\right]^{\frac{1}{2}}$$

$$\frac{1}{w_j} = \frac{3}{4(N_j - 4)}$$

The test statistic (4.5.2) is based on the approximation

$$\frac{2R}{N} \approx N\left[k, \left\{\frac{2(1 - a^2k^2)}{N}\right\}^2\right] \tag{4.5.3}$$

which is obtained from the approximation

$$A(k) \approx \frac{k}{2}\left[1 - \frac{k^2}{8}\right]$$

from (3.2.8).

The functional form of the transformation to normality is

$$g_1(k) = \sin^{-1}(ak)$$

under H_0 (4.5.1) the $g_1(R_j)$ are approximately distributed as independent $N(g_1(k),\sigma_j)$ where

$$\sigma_j^2 = \frac{3}{4(N_j - 4)} = \frac{1}{w_j}$$

4.5.2 $\quad 0.45 \leqslant R_j/N_j \leqslant 0.70$

Test Statistic

$$U_2 = \sum_{j=1}^{q} w_j g_2^2\left[\frac{R_j}{N_j}\right] - \frac{\left\{\sum_{j=1}^{q} w_j g_2\left[\frac{R_j}{N_j}\right]\right\}^2}{\sum_{j=1}^{q} w_j} = \chi^2_{q-1} \tag{4.5.4}$$

where

$$\frac{1}{w_j} = \frac{0.7979}{(N_j - 3)}$$

$$g_2\left[\frac{R}{N}\right] = \sinh^{-1}\left\{\frac{(R/N - c_1)}{c_2}\right\} \qquad c_1 = 1.0894 \qquad c_2 = 0.25789$$

Test Statistic (4.5.4) is built in the same manner as for (4.5.2).

4.5.3 $\quad R_j/N_j > 0.70$

This test is the analogue of Bartlett's test for homogeneity of variance and was given by Stephens (1972) and Mardia (1972).

$$\text{Let } d_j = N_j - 1 \qquad D = N-q = \sum_{j=1}^{q} d_j$$

Test Statistic

$$Z_1 = D\{\log_e \sum_{j=1}^{q} (N_j - R_j)\} - D\log_e D - \sum_{j=1}^{q} d_j\log_e(N_j - R_j) + \sum_{j=1}^{q} d_j\log_e d_j \tag{4.5.5}$$

then $U_3 = Z_1/C$ where

$$C = 1 + \frac{\left[\sum_{j=1}^{q} \frac{1}{d_j} - \frac{1}{D}\right]}{3(q - 1)}$$

U_3 is distributed as χ^2_{q-1}

The above tests will be used when techniques in later chapters assume that *k* is equal in value for all the subpopulations being tested under the composite hypothesis.

CHAPTER 5

PARAMETER ESTIMATION LEADING TO MAXIMUM LIKELIHOOD METHODS FOR LARGER EXPERIMENTAL DESIGNS

5.1 Introduction

In the first four chapters of this thesis, a general review and critique of the problems, theory, approximations and tests associated with circular statistics have been discussed. The thesis will now begin to extend these techniques and discuss alternative methods to construct new tests for experimental designs and their required analysis of variance.

Gould (1969) gave a regression analysis procedure for use when the dependent variable is the position of a point on the circumference of a circle or the surface of a sphere. Gould used an analogue of the normal theory of linear regression for the circular variable problem, where

$$\theta_i = \mu_0 + \beta t_i + \epsilon_i \tag{5.1.1}$$

$\theta_i, i=1,\ldots,N$ is independently distributed as $M(\mu_0 + \beta t_i, k)$ where $t_1,\ldots,t_N$ are known numbers while μ_0, β and k are unknown parameters. The maximum likelihood method, as discussed in Chapter 2, is used to estimate the parameters μ_0 and β. The logarithm of the likelihood function is

$$= \text{constant} + k \sum_{i=1}^{N} \cos(\theta_i - \mu_0 - \beta t_i)$$

The maximum likelihood solutions, $\hat{\mu}_0$ and $\hat{\beta}$ for the parameters are then obtained as the solutions to the two equations

$$\sum_{i=1}^{N} \sin(\theta_i - \hat{\mu}_0 - \hat{\beta} t_i) = 0 \tag{5.1.2}$$

and

$$\sum_{i=1}^{N} t_i \sin(\theta_i - \hat{\mu}_0 - \hat{\beta} t_i) = 0 \tag{5.1.3}$$

From (5.1.2)

$$\tan \hat{\mu}_0 = \frac{\sum_{i=1}^{N} \sin(\theta_i - \hat{\beta} t_i)}{\sum_{i=1}^{N} \cos(\theta_i - \hat{\beta} t_i)} \tag{5.1.4}$$

$\hat{\beta}$ is obtained by a straightforward iterative procedure where $\bar{\beta}$ is an initial estimate of β and $\bar{\mu}_0$ the corresponding value of $\hat{\mu}_0$ from (5.1.4), for the iteration

$$\hat{\beta} = \bar{\beta} + \frac{\sum_{i=1}^{N} t_i \sin(\theta_i - \hat{\mu}_0 - \bar{\beta} t_i)}{\sum_{i=1}^{N} t_i^2 \cos(\theta_i - \hat{\mu}_0 - \bar{\beta} t_i)} \tag{5.1.5}$$

From these estimates is developed an appropriate test for β=0. For this model, maximum likelihood estimation coincides with least squares estimation.

Johnson and Wehrly (1978) showed that the most serious drawback to Goulds approach is that the likelihood function has infinitely many large peaks.

In this section a similar modelling approach to Gould will be discussed, however, for experimental design models, as in linear modelling, constraints will be placed on the estimating parameters. The maximum likelihood estimates of the model parameters will be produced for the required hypothesis test for the first time. The attainment of these not only enables hypothesis analysis but greater data appreciation, as will be shown in the following sections.

For the development of experimental models each angular variate in a one-way classification may be made up of some overall mean direction μ_0, some effect due to a particular treatment, β_j, and some random variable representing unassignable (residual) effect, ϵ_{ij}. Hence $\beta_j = (\mu_j - \mu_0)$ where μ_j is the mean direction of the jth population.

The form of the model which is used can be expressed as;

$$\theta_{ij} = \mu_0 + \beta_j + \epsilon_{ij} \tag{5.2.1}$$

As in linear analysis of variance the assumptions on the treatment effects are:

a) the treatment terms add on to the mean direction term rather than, for example, multiplying

b) the treatment effects are constant

c) the observation on one block or unit is unaffected by the treatment applied to other units.

In linear analysis it is usually convenient to choose the constraint on the treatment parameters so that they sum to zero; in circular analysis it is correspondingly convenient to choose the constraint on the angles specifying the treatment parameters so that their sines sum to zero. A proof and simple example shows the need for this constraint; using expression (5.2.1) and assuming zero residual

$$\frac{\sum_{i=1}^{p}\sum_{j=1}^{q} \sin \theta_{ij}}{\sum_{i=1}^{p}\sum_{j=1}^{q} \cos \theta_{ij}} = \frac{\sum_{i=1}^{p}\sum_{j=1}^{q} \sin(\mu_0 + \beta_j)}{\sum_{i=1}^{p}\sum_{j=1}^{q} \cos(\mu_0 + \beta_j)} \tag{5.2.2}$$

$$\tan \bar{\theta} = \frac{\cos \mu_0 \sum_{j=1}^{q} \sin \beta_j + \sin \mu_0 \sum_{j=1}^{q} \cos \beta_j}{\cos \mu_0 \sum_{j=1}^{q} \cos \beta_j - \sin \mu_0 \sum_{j=1}^{q} \sin \beta_j}$$

under the constraint $\sum_{j=1}^{q} \sin \hat{\beta}_j = 0$

$$\tan \bar{\theta} = \frac{\sin \mu_0 \sum_{j=1}^{q} \cos \beta_j}{\cos \mu_0 \sum_{j=1}^{q} \cos \beta_j} = \tan \mu_0 \tag{5.2.3}$$

Example 5.2.1

i) Let $\mu_0 = 100^\circ$ $\sum_{j=1}^{4} \hat{\beta}_j = 0$

and

$\beta_1 = 48.5^\circ$ $\beta_2 = 19.38^\circ$ $\beta_3 = -54.04^\circ$

therefore $\beta_4 = -13.84^\circ$ with zero residual

$\theta_{11} = 148.5^\circ$ $\theta_{12} = 119.38^\circ$ $\theta_{13} = 45.96^\circ$ $\theta_{14} = 86.16^\circ$

Using (1.4.5) to calculate the mean direction gives

$\bar{\theta} = 100.582^\circ$

ii) However under the constraint $\sum_{j=1}^{4} \sin \hat{\beta}_j = 0$

then $\beta_4 = -15.745^\circ$ and $\theta_{14} = 84.255^\circ$

to give $\bar{\theta} = 100^\circ$

In the first example the sample mean direction differs from the stated population mean direction when zero residual effect is assumed. When the constraint is that the sine of the treatments sum to zero, the sample mean equals the stated population mean direction, i.e. $\bar{\theta} = \mu_0$. This must hold for all factor effects in larger circular experimental designs.

Using this notation each observation θ_{ij} is an independent observation from a von Mises distributed population with a mean direction of $\mu_0 + \beta_j$ and whose concentration parameter is k, i.e.

$$\theta_{ij} \approx \mathrm{IVM}(\mu_0 + \beta_j, k) \tag{5.2.4}$$

where q is the number of treatments, and p is the number of observations on each treatment, with the constraint

$$\sum_{j=1}^{q} \sin \hat{\beta}_j = 0 \tag{5.2.5}$$

and IVM is read as 'independently von Mises distributed'.

5.3 One-way Classification

Assuming the θ_{ij}'s (i=1,2...p, j=1,2...q) are independently distributed as $M(\mu_0 + \beta_j, k)$, let us construct a test of:

$$\left.\begin{aligned} H_0 &: \beta_1 = \beta_2 = \ldots\ldots = \beta_q = 0 \\ H_1 &: \text{at least one } \beta_j \neq 0 \end{aligned}\right\} \tag{5.3.1}$$

Let $\hat{\mu}_0$ and $(\hat{\mu}_0, \hat{\beta}_j)$ be the maximum likelihood estimates of μ_0 and μ_0, β_j under H_0 and H_1 respectively. The logarithm of the likelihood function is given by:

$$\log L = \text{constant} + k\sum_{i=1}^{p} \sum_{j=1}^{q} \cos(\theta_{ij} - \mu_0 - \beta_j) \tag{5.3.2}$$

a) Under H_0 the maximum likelihood estimate μ_0 of μ_0 is the solution of:

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \sin(\theta_{ij} - \hat{\mu}_0) = 0 \qquad (5.3.3)$$

Therefore

$$\tan \hat{\mu}_0 = \frac{\sum_{i=1}^{p} \sum_{j=1}^{q} \sin \theta_{ij}}{\sum_{i=1}^{p} \sum_{j=1}^{q} \cos \theta_{ij}} = \tan \bar{\theta}.. \qquad (5.3.4)$$

Hence $\hat{\mu}_0$, under H_0, is the overall sample mean direction, $\bar{\theta}$..

b) Under H_1 the maximum likelihood estimates $\hat{\mu}_0$ and $\hat{\beta}_j$ are the solution of:

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \sin(\theta_{ij} - \hat{\mu}_0 - \hat{\beta}_j) = 0 \qquad (5.3.5)$$

and

$$\sum_{i=1}^{p} \sin(\theta_{ij} - \hat{\mu}_0 - \hat{\beta}_j) = 0 \quad \text{'q' equations} \qquad (5.3.6)$$

under the constraint $\sum_{j=1}^{q} \sin \hat{\beta}_j = 0$

Let $A_j = \hat{\mu}_0 + \hat{\beta}_j$

Substituting A_j into (5.3.6)

$$\sum_{i=1}^{p} \sin(\theta_{ij} - A_j) = 0 \qquad (5.3.7)$$

From (1.4.9) A_j is the mean direction of the jth block i.e. $\bar{\theta}_{.j}$.

Therefore we may solve for $\hat{\mu}_0$, $\hat{\beta}_1$, $\hat{\beta}_2$, ..., $\hat{\beta}_q$ from the q equations:

$$\left.\begin{array}{l} \bar{\theta}_{\cdot 1} = \hat{\mu}_0 + \hat{\beta}_1 \\ \bar{\theta}_{\cdot 2} = \hat{\mu}_0 + \hat{\beta}_2 \\ \quad \cdot \quad\; \cdot \quad\; \cdot \\ \quad \cdot \quad\; \cdot \quad\; \cdot \\ \quad \cdot \quad\; \cdot \quad\; \cdot \\ \bar{\theta}_{\cdot q} = \hat{\mu}_0 + \hat{\beta}_q \end{array}\right\} \tag{5.3.8}$$

under the constraint $\sum_{j=1}^{q} \sin \hat{\beta}_j = 0$

Taking the sine and summing the q equations of (5.3.8) produces

$$\sum_{j=1}^{q} \sin \bar{\theta}_{\cdot j} = \sin \hat{\mu}_0 \left[\sum_{j=1}^{q} \cos \hat{\beta}_j \right] + \cos \hat{\mu}_0 \left[\sum_{j=1}^{q} \sin \hat{\beta}_j \right]$$

under the given constraint

$$\sum_{j=1}^{q} \sin \bar{\theta}_{\cdot j} = \sin \hat{\mu}_0 \left[\sum_{j=1}^{q} \cos \hat{\beta}_j \right] \tag{5.3.9}$$

Taking the cosine and summing the q equations of (5.3.8) produces

$$\sum_{j=1}^{q} \cos \bar{\theta}_{\cdot j} = \cos \hat{\mu}_0 \left[\sum_{j=1}^{q} \cos \hat{\beta}_j \right] \tag{5.3.10}$$

Dividing equation (5.3.9) by (5.3.10) gives

$$\frac{\sum_{j=1}^{q} \sin \bar{\theta}_{\cdot j}}{\sum_{j=1}^{q} \cos \bar{\theta}_{\cdot j}} = \frac{\sin \hat{\mu}_0}{\cos \hat{\mu}_0}$$

Therefore

$$\hat{\mu}_0 = \tan^{-1}\left[\frac{\sum_{j=1}^{q} \sin \bar{\theta}_{.j}}{\sum_{j=1}^{q} \cos \bar{\theta}_{.j}}\right] \qquad (5.3.11)$$

Hence, under H_1, $\hat{\mu}_0$ is the circular mean of the q equations, $\bar{\theta}_{.j}$.

The $\hat{\beta}_j$ treatment effects are calculated directly from the q equations of (5.3.8), given $\hat{\mu}_0$ found by (5.3.11) i.e.

$$\hat{\beta}_j = \bar{\theta}_{.j} - \hat{\mu}_0 \qquad (5.3.12)$$

Unlike linear analysis the maximum likelihood of the overall mean direction, μ_0, alters depending on the hypothesis test. Example 5.3.1 shows the parameter estimation and the following hypothesis test at work for a hypothetical data set concerning animal orientations.

<u>Example 5.3.1</u>

In an orientation experiment four samples of animals were observed: one was a control group, the other three were experimental groups. Following treatment, their direction of movement was noted and reproduced in Table 5.3.1.

Table 5.3.1

Control Group	Experimental Group 1	Experimental Group 2	Experimental Group 3
156°	168°	137°	214°
111°	76°	184°	155°
174°	102°	222°	129°
140°	137°	163°	153°
213°	184°	236°	125°
121°	112°	133°	228°
200°	62°	161°	185°
166°	133°	193°	176°

Statistic;

	Resultant $R_{.j}$	Sample Mean Direction $\bar{\theta}_{.j}$	$\hat{k}$
Control Group	6.7031	159.9322°	3.415
Experimental Group 1	6.231	121.5483°	2.622
Experimental Group 2	6.5938	177.9278°	3.182
Experimental Group 3	6.5938	169.9278°	3.182

Overall sample mean direction, $\bar{\theta}_{..} = 158.3153°$

Overall resultant length, $R_{..} = 24.362$

Concentration parameter estimate $\hat{k} = 2.464$

Circular mean of the individual group mean directions $\bar{\bar{\theta}} = 157.8013°$

Each of the sample populations have been tested by Watsons U_n^2 statistic (1961), using the critical values supplied by Stephens (1964), to show von Mises distributed data sets. Similarly, the homogeneity of the concentration parameters have been tested and validated via test statistic (4.5.5).

Parameter Estimates;

Under H_0 of (5.3.1) $\quad \hat{\mu}_0 = 158.3153^\circ$ (from (5.3.4))

Under H_1 of (5.3.1) $\quad \hat{\mu}_0 = 157.8013^\circ$ (from (5.3.11))

$$\left.\begin{aligned}
\hat{\beta}_1 &= \bar{\theta}_{.1} - \hat{\mu}_0 = 2.1309^\circ \\
\hat{\beta}_2 &= \bar{\theta}_{.2} - \hat{\mu}_0 = -36.253^\circ \\
\hat{\beta}_3 &= \bar{\theta}_{.3} - \hat{\mu}_0 = 20.1265^\circ \\
\hat{\beta}_4 &= \bar{\theta}_{.4} - \hat{\mu}_0 = 12.1265^\circ
\end{aligned}\right\} \quad \sum_{j=1}^{q} \sin \hat{\beta}_j = 0$$

Hypothesis Testing;

The likelihood ratio test for H_0 and H_1 gives the test statistic

$$N \log_e\left[\frac{I_0(\hat{k}_1)}{I_0(\hat{k}_0)}\right] + \hat{k}_0 \sum_{i=1}^{p} \sum_{j=1}^{q} \cos(\theta_{ij} - \hat{\mu}_0)$$

$$- \hat{k}_1 \sum_{i=1}^{p} \sum_{j=1}^{q} \cos(\theta_{ij} - \hat{\mu}_0 - \hat{\beta}_j) \tag{5.3.13}$$

which produces the same test statistic given by Watson and Williams and built in Chapter 4.3 where

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \cos(\theta_{ij} - \hat{\mu}_0) = R_{..}$$

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \cos(\theta_{ij} - \hat{\mu}_0 - \hat{\beta}_j) = \sum_{j=1}^{q} R_{.j}$$

Using the approximations (3.3.7) and (3.6.7) for large k the one-way classification test statistic is seen as

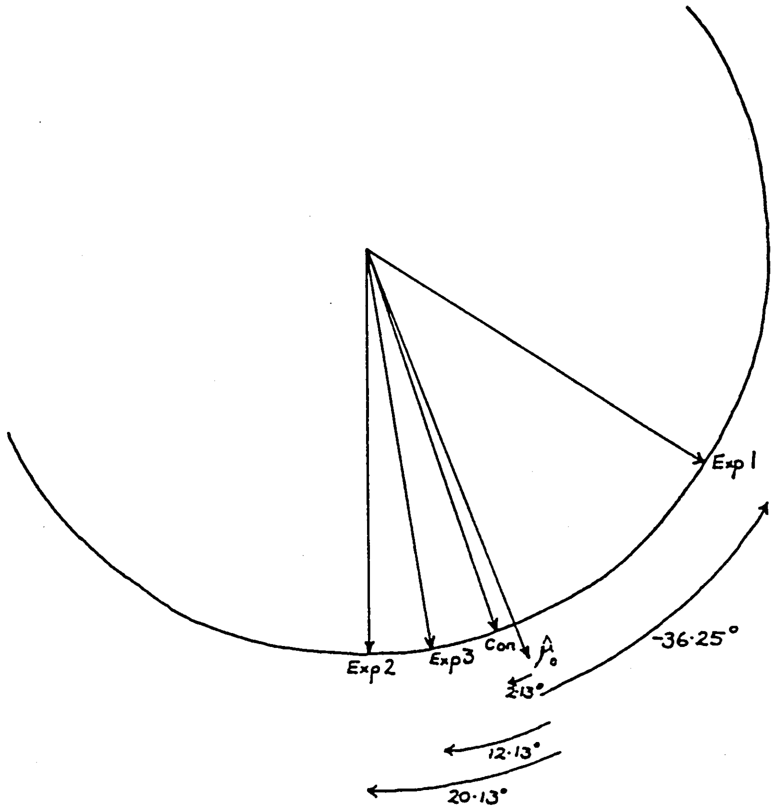
$$N \log_e \left\{ \frac{N - R_{..}}{N - \sum_{j=1}^{q} R_{.j}} \right\} \approx \chi^2_{q-1} \qquad (5.3.14)$$

which is a monotonic function of the F statistic given by (4.4.5). For Example 5.3.1, applying test statistic (5.3.14) gives:

$$32 \log_e \left\{ \frac{7.638}{5.8783} \right\} = 8.38 \approx \chi^2_3$$

$$\chi^2_3 \ (5\%) = 7.81 \qquad \chi^2_3 \ (1\%) = 11.34$$

Indicating a significant difference, at the 5% level, between the four treatments shown in Table 5.3.1 . Having rejected H_0 the above method has shown means by which the parameter estimates of a one-way classification design may be obtained and examined further. Figure 5.3.1 illustrates the angular difference between the four effects or treatments, given in Example 5.3.1, against the model derived mean direction estimate, $\hat{\mu}_0$. Treatment 1 is seen to possess the greatest divergence from $\hat{\mu}_0$.



<u>Figure 5.3.1</u> Angular differences between the treatment mean directions and the overall sample mean direction found in Example 5.3.1 .

5.4 Randomised Complete Block and Larger Experimental Designs

If the maximum likelihood estimates for other circular experimental design models can be found in a similar manner to the one-way classification shown above, the testing of differing hypotheses for 'larger' experimental designs may be undertaken. However, unlike the one-way classification, the parameter estimates for these designs may not be found by simple algebraic manipulation.

Let us investigate, for example, parameter estimation for the randomised complete block design. Here the form of the model may be expressed as

$$\theta_{ij} = \mu_0 + \alpha_i + \beta_j + \varepsilon_{ij} \tag{5.4.1}$$

Assuming the θ_{ij}'s (i=1,2,.....,p, j=1,2,.....,q) are independently distributed as $M(\mu_0+\alpha_i+\beta_j,\ k)$, let us test the 'column' effects for the hypothesis given in (5.3.1) under the constraints:

$$\sum_{i=1}^{p} \sin \hat{\alpha}_i = 0 \tag{5.4.2}$$

$$\sum_{j=1}^{q} \sin \hat{\beta}_j = 0 \tag{5.4.3}$$

let $(\hat{\mu}_0, \hat{\alpha}_i)$ and $(\hat{\mu}_0^*,\ \hat{\alpha}_i^*,\ \hat{\beta}_j^*)$ be the maximum likelihood estimates under H_0 and H_1 respectively. (The starred parameters represent parameter estimates found under H_1).

Under H_0 the maximum likelihood estimates $\hat{\mu}_0$ and $\hat{\alpha}_i$ of μ_0 and α_i are the solutions of:

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \sin(\theta_{ij} - \hat{\mu}_0 - \hat{\alpha}_i) = 0 \tag{5.4.4}$$

$$\sum_{j=1}^{q} \sin(\theta_{ij} - \hat{\mu}_0 - \hat{\alpha}_i) = 0 \quad \text{'p' equations} \tag{5.4.5}$$

under the constraint $\sum_{i=1}^{p} \sin \hat{\alpha}_i = 0$

Equations (5.4.4) and (5.4.5) are in precisely the same form as (5.3.5) and (5.3.6), and therefore give the same parameter estimates as (5.3.12) and (5.3.11) for $\hat{\alpha}_i$ and $\hat{\mu}_0$ respectively.

Under H_1 the maximum likelihood estimates $\hat{\mu}_0^*$, $\hat{\alpha}_i^*$ and $\hat{\beta}_j^*$ of μ_0, α_i and β_j are the solutions of

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \sin(\theta_{ij} - \hat{\mu}_0^* - \hat{\alpha}_i^* - \hat{\beta}_j^*) = 0 \qquad (5.4.6)$$

$$\sum_{i=1}^{p} \sin(\theta_{ij} - \hat{\mu}_0^* - \hat{\alpha}_i^* - \hat{\beta}_j^*) = 0 \qquad \text{'q' equations} \qquad (5.4.7)$$

under the constraint $\sum_{i=1}^{p} \sin \hat{\alpha}_i^* = 0$

$$\sum_{j=1}^{q} \sin(\theta_{ij} - \hat{\mu}_0^* - \hat{\alpha}_i^* - \hat{\beta}_j^*) = 0 \qquad \text{'p' equations} \qquad (5.4.8)$$

under the constraint $\sum_{j=1}^{q} \sin \hat{\beta}_j^* = 0$

Unlike the one-way classification of section 5.3, these constrained equations may not be simplified in the same manner. In 'linear' statistical analysis the equivalent expressions would simplify to their respective 'column' or 'row' means. This produces a simple statement where, some overall mean plus a particular 'row' effect gives the respective row mean; and similarly for column effects. In circular analysis these expressions will not simplify and require lagrange multipliers to take account of the given constraints.

For the solution of the system of equations (5.4.6), (5.4.7) and (5.4.8), under the constraints (5.4.2) and (5.4.3), computer programs have been utilised. Several algorithms were applied to the constrained optimisation problems in an attempt to find a global maximum subject to equality constraints. Of main use were sequential lagrangian methods with the maximisation being solved by quasi-Newton procedures. The functions, however, are not unimodal in nature and are indeed very heavily multimodal. Therefore unless the initial estimates are very good approximations, the algorithms simply find local constrained maximums and not the required global maximum.

In order to solve merely small randomised block designs a lengthy and inefficient program has been written using a direct search method. Example 5.4.1 gives a small hypothetical data set on which the direct search program has been applied.

Example 5.4.1

		'Column' Effects			Resultant	Mean Direction
		β_1	β_2	β_3	$R_{i.}$	$\bar{\theta}_{i.}$
'Row' Effects	α_1	115°	105°	20°	2.229	83.347
	α_2	170°	180°	65°	1.899	145.343
	α_3	75°	90°	325°	1.761	52.253
	α_4	120°	150°	45°	2.175	107.632
Resultant $R_{.j}$		3.346	3.255	3.202		
Mean Direction $\bar{\theta}_{.j}$		119.517°	130.751°	25.566°		

$N = 12$

$R_{..} = 6.817 \quad \bar{\theta}_{..} = 97.458°$

$H_0 : \beta_1 = \beta_2 = \beta_3 = 0$

H_1 : at least one $\beta_j \neq 0$

Parameter Estimation;

$$\text{Under } H_0 : \hat{\mu}_0 = \arctan \left\{ \frac{\sum_{i=1}^{p} \sin \overline{\theta}_{i.}}{\sum_{i=1}^{p} \cos \overline{\theta}_{i.}} \right\} = 96.856^\circ$$

$$\left.\begin{aligned}
\hat{\alpha}_1 &= \overline{\theta}_{1.} - \hat{\mu}_0 = -13.508^\circ \\
\hat{\alpha}_2 &= \overline{\theta}_{2.} - \hat{\mu}_0 = 48.487^\circ \\
\hat{\alpha}_3 &= \overline{\theta}_{3.} - \hat{\mu}_0 = -44.603^\circ \\
\hat{\alpha}_4 &= \overline{\theta}_{4.} - \hat{\mu}_0 = 10.776^\circ
\end{aligned}\right\} \sum_{i=1}^{p} \sin \hat{\alpha}_i = 0$$

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \cos(\theta_{ij} - \hat{\mu}_0 - \hat{\alpha}_i) = \sum_{i=1}^{p} R_{i.} = 8.0646$$

$\hat{k}$ using approximation (3.2.7) equals 1.838

Under H_1 : Via the direct search method

$$\hat{\mu}_0{}^* = 97.065^\circ$$

$$\left.\begin{aligned}
\hat{\alpha}_1{}^* &= -11.78^\circ \\
\hat{\alpha}_2{}^* &= 46.49^\circ \\
\hat{\alpha}_3{}^* &= -48.49^\circ \\
\hat{\alpha}_4{}^* &= 13.164^\circ
\end{aligned}\right\} \sum_{i=1}^{p} \sin \hat{\alpha}_i{}^* = 0$$

$$\left.\begin{aligned}
\hat{\beta}_1{}^* &= 23.13^\circ \\
\hat{\beta}_2{}^* &= 34.35^\circ \\
\hat{\beta}_3{}^* &= -73.15^\circ
\end{aligned}\right\} \sum_{j=1}^{q} \sin \hat{\beta}_j{}^* = 0$$

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \cos(\theta_{ij} - \hat{\mu}_0{}^* - \hat{\alpha}_i{}^* - \hat{\beta}_j{}^*) = 11.878$$

Maximum likelihood estimate $\hat{k}_1$ is given by

$$\frac{I_1(\hat{k}_1)}{I_0(\hat{k}_1)} = \frac{1}{N} \sum_{i=1}^{p} \sum_{j=1}^{q} \cos(\theta_{ij} - \hat{\mu}_0^* - \hat{\alpha}_i^* - \hat{\beta}_j^*)$$

Using approximation (3.3.11) $\hat{k}_1 = 49.4$

Replacing the parameter estimates under H_0 and H_1 into the likelihood ratio test statistic (5.3.13) and using the approximation (3.6.7) produces a chi-squared value of 41.88. Comparing this to a χ^2_3 indicates that there is a highly significant difference between the column or block effects.

Under a null hypothesis testing the row or treatment effects the m.l.e. of $\hat{\mu}_0$ and $\hat{\beta}_j$ are

$$\hat{\mu}_0 = \arctan \left\{ \frac{\sum_{j=1}^{q} \sin \bar{\theta}_{.j}}{\sum_{j=1}^{q} \cos \bar{\theta}_{.j}} \right\} = 96.74^\circ$$

$$\left.\begin{aligned} \hat{\beta}_1 &= \bar{\theta}_{1.} - \hat{\mu}_0 = 22.777^\circ \\ \hat{\beta}_2 &= \bar{\theta}_{2.} - \hat{\mu}_0 = 34.011^\circ \\ \hat{\beta}_3 &= \bar{\theta}_{3.} - \hat{\mu}_0 = -71.174^\circ \end{aligned}\right\} \quad \sum_{j=1}^{q} \sin \hat{\beta}_j = 0$$

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \cos(\theta_{ij} - \hat{\mu}_0 - \hat{\beta}_j) = \sum_{j=1}^{q} R_{.j} = 9.8034$$

$$\hat{k}_0 = 3.01$$

The alternative hypothesis estimates will be unchanged producing a chi-squared value of 34.74. Comparing this to a χ^2_2 indicates a highly significant difference between the row or treatment effects.

5.5 Summary

In linear statistical analysis the generalised linear model approach may be used to find the factor parameter estimates for a particular design. These are then utilized within a maximum likelihood testing procedure to examine for any significant difference between the factor effects. A similar approach has been used within this section for circular statistics. In linear analysis the factor effects sum to zero, here we have seen that for circular analysis, it is convenient to choose the constraint on the angles specifying the treatment parameters so that their sines sum to zero. From this, section 5.3 has shown how parameter estimation for the design model of a one–way classification may be produced, and assist in further understanding the underlying structure of the data sample under investigation.

For larger designs, however, the solution of the likelihood function under the given constraints may not be found by the same simple algebraic manipulation. Due to the complexity of the maximising problems involved many local maxima may be found and the discovery of the global maxima extremely difficult. Many computer programs for the optimisation of constrained equations have been tried with little success and further investigation into improved computer algorithms will be necessary.

If the null hypothesis for a particular test is rejected, the neatness of this approach helps us to appreciate and to look at the contrasts between the effects within a factor. Such procedures exist for linear analysis derived by Tukey or Scheffe known as methods for multiple comparison. If the approach of this section can be extended to larger design methods using computer algorithms to help find and understand the parameter estimates, similar procedures of multiple comparison may be derived for the circular case.

CHAPTER 6

THE EXTENSION OF EXISTING TECHNIQUES TO LARGER EXPERIMENTAL DESIGNS

6.1 Introduction

From Chapter 4 we have seen how, for tightly clustered populations, Watson and Williams (1956) developed a one-way classification technique which has been widely used for 2 and 3 dimensional vectors. This original technique has been refined and used to produce tests for several samples having common mean direction. Chapter 5 has shown how the parameter estimates may be found for the one-way classification design. In Section 6.2 this analysis of variance is extended from the one-way layout to the nested or hierarchical design as an initial step to the analysis of larger more complex experimental situations. Section 6.3 extends the design further to enable analysis of randomised complete block designs and the two-way analysis of variance design, for large k.

It should be noted that within linear analysis of variance the factor components within an experimental design are correctly referred to as sums of squares, however, in circular analysis of variance the measures of each factor are not calculated in the same manner and will be referred to as measures of variation. For examples, the one-way analysis of variance procedure will be analysed using a total, between and residual measure of variation.

6.2 Nested or Hierarchical Design

There are many occasions when a researcher wishes to study the effect of a single factor such as light intensity or magnetic charge on the displacement of micro-organisms but because of the nature in which these are obtained a more complex one-way design is required. The nested or hierarchical model is such a design.

It is often necessary to measure the response to a treatment on each individual of a subsample of a unit factor than on the entire unit to which the treatment is applied. In a drug experiment, for example, the treatments may have been applied to a particular animal type within different groups, from each group several animals may be picked at random and their response (perhaps their angle of movement) measured. To this end a model for the nested design may be built, under the general assumptions;

i) The samples are drawn from populations with a von Mises distribution

ii) The parameter of concentration has the same value in each population, that is,

$$k_1 = k_2 = \ldots\ldots = k_q = k$$

iii) The overall and individual parameters of concentration are sufficiently large, namely

$$k \geqslant 2$$

Let p be the number of treatments, given by $i = 1,2,\ldots.,p$, q_i the number of groups or experimental units within treatment i, given by $j = 1, 2, \ldots.q_i$, and n the number of observations within each q_i group. Let N be the total number of observations and U be the number of cells. Extending the expression given by Watson and Williams (1956), that

$$2k(N - R_{..}) = 2k(\sum_{i=1}^{p} R_{i.} - R_{..}) + 2k(N - \sum_{i=1}^{p} R_{i.}) \tag{6.2.1}$$

produces:

$$2k(N - R_{...}) = 2k(\sum_{i=1}^{p} R_{i..} - R_{...}) + 2k(\sum_{j=1}^{q_1} R_{1j.} - R_{1..})$$

$$+ 2k(\sum_{j=2}^{q_2} R_{2j.} - R_{2..}) + \ldots\ldots\ldots\ldots\ldots$$

$$+ 2k(\sum_{j=1}^{q_p} R_{pj.} - R_{p..}) + 2k(\sum_{i=1}^{p} \sum_{j=1}^{q_i} n_{ij.} - R_{1j.}) \tag{6.2.2}$$

with associated independent chi-squared distributions with known degrees of freedom, for large k:

$$\chi^2_{N-1} = \chi^2_{p-1} + \chi^2_{q_1-1} + \chi^2_{q_2-1} + \ldots\ldots\ldots\ldots + \chi^2_{q_p-1} + \chi^2_{N-U}$$

The first term on the right hand side of the expression (6.2.2) is essentially the measure of variation among treatments, the next q_p terms of similar form are the measures of variation within treatment i but among the q_i groups. The final term represents the within or residual variation within treatment and within group but among the sub-groups. As a nested design we may consider differences between rows, or difference between columns within any one row.

As in the one-way analyses a test statistic for examining the differences between rows may be given as

$$\frac{(N - U)(\sum_{i=1}^{p} R_{i..} - R_{...})}{(p - 1)(N - \sum_{i=1}^{p} \sum_{j=1}^{q_i} R_{ij.})} \tag{6.2.3}$$

which has an F distribution with (p–1) and (N–U) degrees of freedom. Similarly a test statistic for the difference between columns within row i is calculated in the same manner.

$$\frac{(N - U)(\sum_{j=1}^{q_i} R_{ij.} - R_{i.})}{(q_i - 1)(N - \sum_{i=1}^{p} \sum_{j=1}^{q_i} R_{ij.})} \qquad (6.2.4)$$

which has an F distribution with (q_i-1) and (N–U) degrees of freedom.

Stephens (1982) produced the same expression (6.2.2) and quotients (6.2.3) and (6.2.4) in m dimensions for the analysis of data which are proportions of a continuum such as time or volume. Stephens also gives a good example of the methodology when studying the proportion of time spent in various activities by 130 students.

6.3 The Randomised Complete Block Design and Two-way Classification with Interaction Design

The randomised block design is a widely used method of dealing with factors that are known to be important and which the researcher wishes to eliminate rather than to study. Here the factor is blocked so that each is as homogeneous as possible and the treatments under study are each used exactly once in each block for the design to be balanced. The observed differences among the treatments should be largely unaffected by the factor that has been blocked.

There are many situations where a randomised block plan can be profitably utilised. For example, a testing scheme may take several days to complete. If we expect some systematic differences between days, we might plan to observe each item on each day; a day would then represent a block. Since each treatment occurs exactly once in every block, the treatment totals or means are directly comparable without adjustment.

By again extending the simple expression (6.2.1) and its associated results concerning the chi-squared decomposition we may now take account of a possible block effect, i, as well as the treatment effect, j, to produce the randomised complete block model;

$$2k(N - R_{..}) = 2k(\sum_{i=1}^{p} R_{i.} - R_{..}) + 2k(\sum_{j=1}^{q} R_{.j} - R_{..})$$

$$+ 2k(N - \sum_{i=1}^{p} R_{i.} - \sum_{j=1}^{q} R_{.j} + R_{..}) \qquad (6.3.1)$$

with associated independent chi-squared distributions, for large k,

$$\chi^2_{N-1} = \chi^2_{p-1} + \chi^2_{q-1} + \chi^2_{(p-1)(q-1)}$$

Expression (6.3.1) obeys the χ^2 decomposition property parallel to the linear form of the randomised complete block design. The first term on the right hand side of the expression is essentially a measure of variation due to treatments, the second term of similar form being a measure of variation due to blocks. The final term represents the residual variation after variation due to treatments and blocks have been removed.

From (6.3.1) the test statistic (6.3.2) is produced to test the null hypothesis that there is no difference between the treatments.

$$Z_1 = \frac{(p - 1)(q - 1)(\sum_{i=1}^{p} R_{i.} - R_{..})}{(p - 1)(N - \sum_{i=1}^{p} R_{i.} - \sum_{j=1}^{q} R_{.j} + R_{..})} \qquad (6.3.2)$$

which has an F distribution with (p−1) and (p−1)(q−1) degrees of freedom. Similarly, the test statistic (6.3.3) will be produced to test the null hypothesis that there is no difference between the blocks

$$Z_2 = \frac{(p-1)(q-1)(\sum_{i=1}^{q} R_{.j} - R_{..})}{(q-1)(N - \sum_{i=1}^{p} R_{i.} - \sum_{j=1}^{q} R_{.j} + R_{..})} \tag{6.3.3}$$

which has an F distribution with (q−1) and (p−1)(q−1) degrees of freedom.

In the case of significance, we may only state that the mean directions are or are not equal. The test does not allow for discrimination among single mean directions.

Before the accuracy of the approximations to the distributions of the components of (6.3.1) and the F distribution approximations Z_1 and Z_2 are examined, the associated model for the two-way analysis with interaction will be discussed. A discussion of interaction in directional data analysis will be given in Chapter 8.

Using the same approach as for the nested and randomised complete block designs a two-way classification with interaction may be built;

$$\begin{aligned} 2k(N - R_{...}) &= 2k(\sum_{i=1}^{p} R_{i..} - R_{...}) + 2k(\sum_{j=1}^{q} R_{.j.} - R_{...}) \\ &+ 2k(N - \sum_{i=1}^{p} \sum_{j=1}^{q} R_{ij.}) \\ &+ 2k(\sum_{i=1}^{p} \sum_{j=1}^{q} R_{ij.} - \sum_{i=1}^{p} R_{i..} - \sum_{j=1}^{q} R_{.j.} + R_{...}) \end{aligned} \tag{6.3.4}$$

The first three terms on the right hand side of (6.3.4) have been shown to be chi–squared distributed by Watson and Williams (1956) and Stephens (1963) representing measures of variation due to row i and column j effects and a measure of residual variation. The final term represents the possible interaction within the experiment. F test statistics may be produced in the same manner as for the nested and randomised block designs. (l represents the number of observations on each treatment combination).

Testing row effects

$$Z_3 = \frac{pq(l - 1)(\sum_{i=1}^{p} R_{i..} - R_{...})}{(p - 1)(N - \sum_{i=1}^{p} \sum_{j=1}^{q} R_{ij.})} \tag{6.3.5}$$

$$= F_{(p-1),pq(l-1)}$$

Testing column effects

$$Z_4 = \frac{pq(l - 1)(\sum_{j=1}^{q} R_{.j.} - R_{...})}{(q - 1)(N - \sum_{i=1}^{p} \sum_{j=1}^{q} R_{ij.})} \tag{6.3.6}$$

$$= F_{(q-1),pq(l-1)}$$

Testing interaction effects

$$Z_5 = \frac{pq(l - 1)(\sum_{i=1}^{p} \sum_{j=1}^{q} R_{ij.} - \sum_{i=1}^{p} R_{i..} - \sum_{j=1}^{q} R_{.j.} + R_{...})}{(p - 1)(q - 1)(N - \sum_{i=1}^{p} \sum_{j=1}^{q} R_{ij.})} \tag{6.3.7}$$

$$= F_{(p - 1)(q - 1),pq(l - 1)}$$

6.4 Accuracy of the Associated Component χ^2 Approximations for the Randomised Block Design and Two-way Classification and their Corresponding F Statistics

The expressions (6.3.1) and (6.3.4) are based on a sequence of approximations, and the accuracy of their chi-squared approximations may best be determined by simulation. An examination may be made by taking Monte Carlo samples from a von Mises distribution specified with fixed k. Observations from the distribution specified by the null hypothesis were generated by computer methods outlined in Appendix B and were grouped into samples of size N. This has been carried out for 10,000 sets of samples of various size and were drawn from von Mises distributions with k = 2, 3, 4, 5 and 10. For the testing of the randomised block model three designs were investigated varying in sizes of N. Tables 6.4.1 to 6.4.5 examine the chi-squared approximations for each of the components within each of the models. Table 6.4.1 shows the accuracy for the total measure of variation component, $2k(N-R_{..})$, and Stephens improved approximation, $2\gamma(N-R_{..})$, where γ is given by (4.3.4). It is clearly seen that α_i, the simulated proportion of the component, approaches α, the χ^2 value theoretical proportion or significance level, when γ is applied. This is further illustrated when the first two moments for both approximations are given in Table 6.4.2. When Stephens improvement is used both moments approach their expected values and with increased accuracy as k increases. Accuracy for both approximations increases as k increases.

Similar results are also found when the component of error or residual is also adjusted by Stephens improvement, γ. Table 6.4.3 compares the accuracy of the $\chi^2(p-1)(q-1)$ approximations, and shows that the accuracy for both approximations improves as k increases. Table 6.4.4 shows the effect on the two approximations first two moments when Stephens improvement is applied. As with the total measure of variation component, when γ is applied, both moments approach their expected values with increased accuracy as k increases.

Table 6.4.5 gives the accuracy of the two components measuring block and treatment effects. Comparing the tables of the two effects shows how the number of observations as well as the size of concentration parameter affects the accuracy of the chi–squared approximation. As N and k increase the accuracy of the chi–squared increases.

TABLE 6.4.1 Comparing the accuracy of the χ^2 approximations with N-1 degrees of freedom for the total measure of variation

SAMPLE SIZES	k	$2k(N-R_{..})$ α 0.90	0.95	0.99	$2Y(N-R_{..})$ α 0.90	0.95	0.99
3 by 5	2	0.7517	0.852	0.963	0.8953	0.9525	0.9942
	3	.807	.8873	.9708	.8874	.942	.9885
p = 3 treatments	4	.8408	.9034	.9717	.8885	.94	.9856
q = 5 blocks	5	.8566	.9252	.9824	.9024	.9505	.9904
N = 15	10	.883	.939	.9875	.9008	.9538	.9905
3 by 8	2	0.8834	0.8238	0.951	0.896	0.9514	0.9928
	3	.7743	.8707	.9598	.8911	.9438	.9868
p = 3 treatments	4	.8292	.8986	.9734	.8948	.946	.991
q = 8 blocks	5	.8435	.9158	.9779	.904	.9516	.9885
N = 24	10	.8776	.9394	.988	.9046	.9523	.9926
5 by 10	2	0.5987	0.7261	0.8968	0.879	0.9552	0.991
	3	.7243	.8243	.9396	.882	.938	.9836
p = 5 treatments	4	.7838	.8736	.962	.8977	.9452	.9906
q = 10 blocks	5	.8212	.9008	.9715	.9062	.9523	.9874
N = 50	10	.87	.9298	.9834	.906	.953	.9912

The table gives the proportion of 10,000 Monte Carlo samples for which the two approximations fell below the χ^2_{N-1} value for which the theoretical proportion should be α.

TABLE 6.4.2 Comparing the mean and variance of the χ^2 approximations with N-1 degrees of freedom to their expected values

SAMPLE SIZES	k	2k(N-R$_{..}$) Mean	(Variance)	2Y(N-R$_{..}$) Mean	(Variance)
3 by 5	2	17.119	(36.629)	14.418	(25.963)
	3	16.0	(36.95)	14.225	(29.191)
Expected	4	15.435	(36.717)	14.114	(30.695)
Mean = 14	5	15.063	(32.568)	14.013	(28.179)
Variance = 28	10	14.45	(29.209)	13.929	(27.137)
3 by 8	2	28.065	(59.388)	23.633	(42.061)
	3	26.384	(60.737)	23.453	(47.942)
Expected	4	25.088	(57.857)	22.938	(48.333)
Mean = 23	5	24.677	(54.721)	22.956	(47.341)
Variance = 46	10	23.751	(47.999)	22.893	(44.574)
5 by 10	2	59.64	(135.43)	50.221	(96.619)
	3	55.978	(135.311)	49.756	(106.981)
Expected	4	53.688	(121.712)	49.085	(101.84)
Mean = 49	5	52.175	(116.535)	48.533	(100.934)
Variance = 98	10	50.452	(108.791)	48.628	(101.086)

TABLE 6.4.3 Comparing the accuracy of the χ^2 approximations with $(p-1)(q-1)$ degrees of freedom for the residual measure of variation

SAMPLE SIZES	k	$2k(N-\Sigma R_{i.} - \Sigma R_{.j} + R_{..})$ α			$2y(N-\Sigma R_{i.} - \Sigma R_{.j} + R_{..})$ α		
		0.90	0.95	0.99	0.90	0.95	0.99
3 by 5	2	0.7333	0.839	0.9641	0.854	0.9342	0.9945
	3	.7867	.8712	.967	.8512	.9247	.9856
p = 3 treatments	4	.8288	.9051	.972	.8803	.9326	.9837
q = 5 blocks	5	.8486	.9184	.9791	.8863	.9409	.987
N = 15	10	.8832	.9393	.9876	.9008	.9498	.9908
3 by 8	2	0.6874	0.8105	0.9467	0.8578	0.9287	0.991
	3	.7495	.849	.9548	.8491	.9202	.9815
p = 3 treatments	4	.8166	.8954	.974	.8814	.9363	.9892
q = 8 blocks	5	.8383	.9072	.9767	.8823	.9392	.9865
N = 24	10	.8748	.9486	.986	.8966	.9476	.9898
5 by 10	2	0.556	0.6992	0.8862	0.8355	0.9177	0.9841
	3	.70	.8029	.926	.8439	.9084	.9746
p = 5 treatments	4	.774	.8661	.9599	.8774	.9333	.9864
q = 10 blocks	5	.814	.8926	.9732	.8883	.9461	.9884
N = 50	10	.8602	.9263	.9835	.8958	.9478	.988

The table gives the proportion of 10,000 Monte Carlo samples for which the two approximations fell below the $\chi^2_{(p-1)(q-1)}$ value for which the theoretical proportion should be α.

TABLE 6.4.4 Comparing the mean and variance of the χ^2 approximations with (p-1)(q-1) degrees of freedom to their expected values

SAMPLE SIZES	k	$2k(N-\Sigma R_{i.}-\Sigma R_{.j}+R_{..})$ Mean	(Variance)	$2y(N-\Sigma R_{i.}-\Sigma R_{.j}+R_{..})$ Mean	(Variance)
3 by 5	2	10.656	(21.956)	8.974	(15.568)
	3	9.786	(23.196	8.699	(18.326)
Expected	4	9.2	(21.795)	8.411	(18.217)
Mean = 8	5	8.893	(19.823)	8.272	(17.152)
Variance = 16	10	8.363	(16.818)	8.06	(15.624)
3 by 8	2	18.384	(38.01)	15.484	(26.937)
	3	17.027	(41.302)	15.137	(32.608)
Expected	4	15.839	(36.208)	14.483	(30.265)
Mean = 14	5	15.475	(34.714)	14.397	(30.04)
Variance = 28	10	14.645	(30.328)	14.116	(28.181)
5 by 10	2	46.075	(103.856)	38.798	(73.688)
	3	42.825	(109.238)	38.066	(86.312)
Expected	4	40.473	(92.3)	37.003	(77.205)
Mean = 36	5	38.961	(87.433)	36.242	(75.646)
Variance = 72	10	37.368	(81.812)	36.017	(76.0)

TABLE 6.4.5 Examining the accuracy of the χ^2 approximations for the block and treatment measures of variation

SAMPLE SIZES	k	$2k(\Sigma R_{.j} - R_{..})$ α 0.90	0.95	0.99	$2k(\Sigma R_{i.} - R_{..})$ α 0.90	0.95	0.99
3 by 5	2	0.8847	0.939	0.9868	0.8707	0.9335	0.9854
	3	.8874	.9423	.9887	.888	.9412	.9856
p = 3 treatments	4	.895	.9474	.9867	.886	.9396	.9859
q = 5 blocks	5	.894	.9482	.988	.8887	.9439	.9875
N = 15	10	.8927	.946	.9889	.8974	.9524	.991
3 by 8	2	0.894	0.9452	0.9879	0.8635	0.927	0.9825
	3	.8914	.9468	.989	.8842	.9366	.989
p = 3 treatments	4	.9045	.9503	.9917	.8829	.9388	.9862
q = 8 blocks	5	.898	.9454	.9894	.8868	.9412	.9876
N = 24	10	.9013	.9509	.9895	.8977	.9481	.9918
5 by 10	2	0.9005	0.9552	0.9882	0.872	0.9294	0.9824
	3	.8958	.9426	.9871	.8965	.9526	.9886
p = 5 treatments	4	.9066	.9478	.9874	.8878	.9443	.9887
q = 10 blocks	5	.8902	.9427	.9888	.8932	.9407	.9873
N = 50	10	.9004	.9522	.9913	.8986	.9532	.9908

Tables 6.4.6 and 6.4.7, probably the most important, show the results for the old and new F–approximations. The tables give the proportion α_i of Monte Carlo results less than the F–value for the appropriate statistics when the theoretical proportion should be α. The old F–approximations being Z_1 and Z_2 given by (6.3.2) and (6.3.3) respectively, and the new F–approximations being mZ_1 and mZ_2 where m is given by (6.4.1) and is the statistic due to Stephens improvement of the residual measure of variation.

$$m = 1 + \frac{3}{8k} \tag{6.4.1}$$

The conclusions here are the same as for the F–statistic for one–way analysis given by Stephens (1972). The new tests are clearly very good F–statistics and increase in accuracy as k and N increase. The approximations are very good, even for N as low as 10, and improve quickly for larger values of N.

The chi–squared approximations of (6.3.4) is examined in the same manner as above. Tables 6.4.8, 6.4.9 and 6.4.10 show the chi–squared approximations for all five components of (6.3.4). The total and residual measures have been adjusted by Stephens improvements, γ, the original approximations of the total and residual measures were of similar form to those already seen in the randomised block design.

All five approximations, including the interaction measure built in section 6.2, show excellent chi–squared approximations for $k \geqslant 2$. Accuracy is seen to improve as k increases.

Tables 6.4.11, 6.4.12 and 6.4.13 give the results for the old and new F–approximations for testing the two main effects and the interaction component. The old F–approximations being Z_3, Z_4, and Z_5 given by (6.3.5), (6.3.6) and (6.3.7), and the new F–approximations given by mZ_3, mZ_4, and mZ_5. Clearly all three F–statistics improve when Stephens improvement is used, similarly and as for the randomised block design, increased accuracy is seen as k and N increase.

TABLE 6.4.6 Comparison of the two F approximations Z_1 and mZ_1

SAMPLE SIZES	k	Z_1 Equation (6.3.2) α 0.90	0.95	0.99	mZ_1 α 0.90	0.95	0.99
3 by 5	2	0.9392	0.9747	0.996	0.9134	0.9627	0.9945
	3	.9269	.9691	.995	.9095	.9572	.9914
	4	.919	.967	.995	.905	.955	.9935
	5	.9124	.9574	.9913	.8997	.952	.9898
	10	.9041	.9553	.9936	.8966	.9517	.9926
3 by 8	2	.9435	0.976	0.9967	0.9156	0.963	0.9939
	3	.9258	.968	.9942	.9034	.9547	.9908
	4	.9234	.9647	.995	.9057	.9574	.993
	5	.9138	.9568	.9927	.901	.9506	.9914
	10	.8749	.9346	.986	.9012	.9528	.9895
5 by 10	2	0.9611	0.9812	0.9963	0.9278	0.9676	0.9926
	3	.9397	.9715	.9942	.9041	.9556	.9909
	4	.9256	.9654	.9937	.9044	.9526	.991
	5	.9172	.957	.9935	.8964	.9455	.9891
	10	.9104	.9588	.9942	.90	.9525	.9918

TABLE 6.4.7 Comparison of the two F approximations Z_2 and mZ_2

SAMPLE SIZES	k	Z_2 Equation (6.3.3) α 0.90	0.95	0.99	mZ_2 α 0.90	0.95	0.99
3 by 5	2	0.951	0.9807	0.9985	0.919	0.9676	0.9948
	3	.935	.9686	.9948	.9125	.9598	.9923
	4	.9197	.9603	.9937	.9072	.9499	.9914
	5	.913	.9584	.9923	.90	.9486	.9905
	10	.911	.9572	.992	.9029	.9537	.9916
3 by 8	2	0.9503	0.9784	0.9952	0.9135	0.9606	0.9906
	3	.9336	.97	.9938	.9054	.9541	.9899
	4	.9271	.9669	.9939	.9054	.957	.9894
	5	.9178	.961	.994	.8992	.949	.991
	10	.8775	.9394	.988	.8976	.951	.9907
5 by 10	2	0.9561	0.9822	0.9969	0.9092	0.953	0.9927
	3	.9482	.977	.9972	.9124	.9613	.992
	4	.9333	.9707	.9965	.9057	.953	.9926
	5	.9192	.961	.9918	.899	.9493	.987
	10	.9188	.958	.9926	.9063	.9518	.991

TABLE 6.4.8 Examining the accuracy of the χ^2 approximations for the total and first main effect measures of variation

SAMPLE SIZES	k	2Y(N-R...) α 0.90	0.95	0.99	$2k(\Sigma R_{i..} - R...)$ α 0.90	0.95	0.99
2 by 3 (by 5)	2	0.8915	0.9486	0.9932	0.8955	0.9483	0.99
	3	.8824	.9402	.988	.8997	.9495	.9896
(2 by 3 design with	4	.8972	.9472	.9912	.8984	.9507	.9913
5 observation within	5	.9037	.952	.9891	.8924	.9458	.988
each cell) N=30	10	.906	.9551	.9894	.904	.9526	.9905
2 by 3 (by 10)	2	0.8844	0.944	0.9897	0.8978	0.9466	0.9902
	3	.8787	.936	.985	.8954	.9491	.9913
N = 60	4	.8972	.9467	.987	.8952	.9458	.9902
	5	.9048	.9582	.99	.8993	.9503	.9899
	10	.912	.9542	.9895	.902	.9514	.9902
3 by 3 (by 5)	2	0.894	0.9497	0.992	0.904	0.949	0.9903
	3	.8914	.9421	.9888	.9058	.9535	.9912
N = 45	4	.897	.9476	.9916	.9048	.954	.988
	5	.8981	.9488	.9907	.8956	.9476	.9901
	10	.9016	.9554	.99	.9036	.9523	.9914
3 by 3 (by 10)	2	0.875	0.9376	0.9891	0.8962	0.9483	0.99
	3	.8795	.9343	.9858	.9046	.9554	.989
N = 90	4	.8906	.945	.9903	.9065	.9515	.9903
	5	.9012	.951	.9909	.8972	.9453	.9882
	10	.9086	.956	.9927	.9012	.9492	.99

TABLE 6.4.9 Examining the accuracy of the χ^2 approximations for the second main effect and interaction measures of variation

SAMPLE SIZES	k	$2k(\Sigma R_{.j.} - R_{...})$ α 0.90	0.95	0.99	$2k(\Sigma\Sigma R_{ij.} - \Sigma R_{i..} - \Sigma R_{.j.} + R_{...})$ α 0.90	0.95	0.99
2 by 3 (by 5)	2	0.904	0.952	0.991	0.8773	0.9336	0.9832
	3	.8918	.9432	.989	.8938	.9492	.9901
(2 by 3 design	4	.8965	.9463	.9918	.8863	.9455	.9896
with 5 observa-	5	.9025	.948	.989	.888	.946	.990
tions within	10	.903	.950	.99	.8962	.9536	.989
each cell)N=30							
2 by 3 (by 10)	2	0.9002	0.946	0.9894	0.8968	0.9492	0.9896
	3	.9053	.9498	.9885	.8932	.9427	.9891
N=60	4	.8974	.9473	.9882	.897	.9484	.989
	5	.8972	.9474	.9915	.898	.9502	.9881
	10	.9038	.9552	.9895	.9007	.95	.9914
3 by 3 (by 5)	2	0.8927	0.9468	0.9906	0.8739	0.9294	0.9404
	3	.8962	.9508	.9914	.9462	.941	.9852
N=45	4	.8955	.9453	.9903	.8854	.9367	.9853
	5	.8994	.9486	.9898	.8974	.9486	.9885
	10	.8914	.9466	.9902	.8992	.948	.9884
3 by 3 (by 10)	2	0.896	0.948	0.989	0.8869	0.9397	0.9847
	3	.8908	.9451	.9884	.8866	.9393	.9865
N=90	4	.9024	.949	.99	.8976	.9483	.9914
	5	.8965	.949	.9883	.8964	.9486	.9912
	10	.898	.9514	.9884	.898	.9498	.9888

TABLE 6.4.10 Examining the accuracy of the χ^2 approximation for the residual measure of variation

SAMPLE SIZES	k	$2Y(N-\Sigma\Sigma R_{ij.})$ α 0.90	0.95	0.99
2 by 3 (by 5)	2	0.8753	0.9386	0.9918
	3	.8688	.9316	.985
(2 by 3 design	4	.889	.9431	.9893
with 5 observa-	5	.8945	.9499	.9881
tions within	10	.9042	.9508	.988
each cell) N=30				
2 by 3 (by 10)	2	0.871	0.9348	0.9875
	3	.8687	.931	.9845
N=60	4	.8906	.944	.9868
	5	.9006	.9498	.9908
	10	.9042	.9537	.988
3 by 3 (by 5)	2	0.8675	0.9423	0.991
	3	.871	.9323	.9863
N=45	4	.8876	.9406	.989
	5	.8993	.944	.9884
	10	.8942	.9495	.9899
3 by 3 (by 10)	2	0.8578	0.925	0.9855
	3	.9633	.9279	.9835
N=90	4	.875	.9372	.9874
	5	.8971	.948	.9891
	10	.9082	.9562	.9914

TABLE 6.4.11 Comparison of the two F approximations Z_3 and mZ_3 for the second main effect in the two-way classification

SAMPLE SIZES	K	Z_3 Equation (6.3.5) α			mZ_3 α		
		0.90	0.95	0.99	0.90	0.95	0.99
2 by 3 (by 5)	2	0.937	0.9712	0.996	0.912	0.958	0.993
	3	.913	.9635	.9946	.8942	.9502	.9906
N = 30	4	.9133	.9618	.9925	.8997	.95	.9912
	5	.9168	.9574	.9934	.9045	.9505	.9907
	10	.905	.9538	.9912	.9002	.9497	.9907
2 by 3 (by 10)	2	0.9338	0.9656	0.9954	0.9071	0.9518	0.9912
	3	.9248	.962	.9931	.9065	.95	.9903
N = 60	4	.9097	.9572	.992	.8957	.949	.9896
	5	.9081	.9591	.9933	.895	.9519	.9913
	10	.9092	.9564	.9916	.9022	.9518	.9908
3 by 3 (by 5)	2	0.9354	0.9764	0.9972	0.9034	0.9561	0.9945
	3	.9273	.9716	.9947	.9022	.9552	.9906
N = 45	4	.916	.9624	.9913	.8991	.9512	.988
	5	.9164	.9601	.9938	.9027	.9511	.99
	10	.901	.9531	.991	.8922	.946	.99
3 by 3 (by 10)	2	0.938	0.9724	0.996	0.906	0.953	0.9892
	3	.9231	.9632	.993	.8962	.9478	.9888
N = 90	4	.9214	.9603	.993	.9038	.949	.9907
	5	.9091	.9587	.9923	.8947	.95	.9902
	10	.9074	.95	.9914	.8987	.9452	.9903

TABLE 6.4.12 Comparison of the F approximations Z_4 and mZ_4 for the first main effect in the two-way classification

SAMPLE SIZES	k	Z_4 Equation (6.3.6) α 0.90	0.95	0.99	mZ_4 α 0.90	0.95	0.99
2 by 3 (by 5)	2	0.9403	0.975	0.9958	0.9102	0.9558	0.9924
	3	.9288	.9665	.9947	.9028	.955	.9928
N = 30	4	.9207	.9602	.9916	.9061	.9514	.9882
	5	.9124	.9549	.9916	.8996	.946	.989
	10	.907	.9586	.9921	.8968	.9528	.9912
2 by 3 (by 10)	2	0.9407	0.9724	0.9977	0.9082	0.953	0.9926
	3	.9249	.9681	.9956	.8981	.9537	.9914
N = 60	4	.9154	.9628	.9922	.897	.9502	.9887
	5	.9154	.96	.993	.9002	.9505	.9907
	10	.9084	.9587	.9923	.9018	.954	.992
3 by 3 (by 5)	2	0.9442	0.9764	0.9952	0.9158	0.96	0.9915
	3	.9334	.9692	.9954	.91	.9556	.993
N = 45	4	.922	.9634	.9945	.906	.9514	.99
	5	.9156	.9614	.9927	.90	.9518	.99
	10	.9126	.9577	.993	.9013	.9535	.9917
3 by 3 (by 10)	2	0.9393	0.9738	0.9958	0.9062	0.9532	0.9907
	3	.9372	.972	.9952	.9124	.9585	.9916
N = 90	4	.926	.9631	.9945	.9083	.954	.99
	5	.912	.9566	.992	.898	.948	.9891
	10	.9096	.9534	.9934	.9026	.9473	.9912

TABLE 6.4.13 Comparison of the two F approximations Z_5 and mZ_5 for the interaction effect in the two-way classification

SAMPLE SIZES	k	Z_5 Equation (6.3.7) α			mZ_5 α		
		0.90	0.95	0.99	0.90	0.95	0.99
2 by 3 (by 5)	2	0.9256	0.9671	0.9945	0.8947	0.9419	0.9887
	3	.9286	.9674	.9943	.908	.9524	.9912
N = 30	4	.911	.9567	.993	.8912	.9442	.989
	5	.9118	.955	.995	.8956	.9446	.9905
	10	.9055	.9518	.9892	.8982	.9466	.9884
2 by 3 (by 10)	2	0.943	0.9764	0.9966	0.9094	0.9573	0.9919
	3	.9226	.9647	.995	.90	.9488	.9908
N = 60	4	.9132	.9598	.9923	.8946	.9475	.9882
	5	.9147	.96	.9916	.8985	.9498	.9891
	10	.9122	.9549	.9914	.9018	.951	.9908
3 by 3 (by 5)	2	0.9291	0.9714	0.9948	0.8898	0.9452	0.9896
	3	.922	.9642	.9942	.892	.9458	.9903
N = 45	4	.9092	.9562	.99	.8857	.9413	.9852
	5	.911	.9607	.9908	.8895	.95	.9875
	10	.9122	.9548	.9914	.8928	.9516	.9908
3 by 3 (by 10)	2	0.9437	0.9759	0.9969	0.9021	0.9504	0.9914
	3	.9238	.9646	.9936	.893	.9462	.9888
N = 90	4	.9245	.9679	.995	.8972	.9525	.9929
	5	.9186	.961	.9923	.8987	.948	.9877
	10	.9078	.9584	.9918	.8936	.9512	.9904

6.5 Summary

As discussed in Chapter 5 the likelihood ratio function has infinitely many equally large peaks and may not be broken down under the set hypotheses using the parameter constraints, as is the case for linear analysis. Because of this the tests derived initially by Watson and Williams and improved by Stephens have been extended, for large k, to enable analysis of further experimental designs. The components within each of the model expressions are shown to be very good chi–squared approximations and their appropriate quotients to be excellent F–distributed approximations, both increasing in accuracy as N and k increase.

Simple extensions to the likelihood criterion for the one–way classification with small concentration parameter showed the test components to be poor chi–squared approximations with little justification for their construction.

CHAPTER 7

THE ROBUSTNESS AND POSSIBLE COLLAPSE OF THE EXTENDED TECHNIQUES

7.1 Introduction

Chapter 6 has shown how we may extend the original techniques for the one–way classification with large k, to examine further experimental designs. This chapter examines the robustness of the test statistics and shows the sensitivity of the assumptions and possible flaws or breakdowns within the new techniques. Section 7.3 gives further understanding to the construction of Watson and Williams test statistic and the reasons for its possible failure when larger designs are considered.

7.2 Robustness of Assumptions

Section 6.2 gave the assumptions which must be observed in order that the new extended techniques may be applied. The third assumption, that the overall concentration parameter is sufficiently large, namely $k \geqslant 2$, is an extremely restrictive assumption when analysing larger designs. Equation (1.4.16) gave an angular measure equivalent to the standard deviation in linear statistics. In order that the assumption of large overall concentration parameter may be satisfied, the measure of angular deviation must not exceed 44° i.e. the overall sample data set must be closely packed. This will not be the case if substantial differences result between or within factors.

Further and more important problems are noted as the size of the concentration parameter decreases towards and below 2. During the many thousands of simulation runs discussed in Section 6.3 many other statistics were collected for each of the varying hypothesised models. These included the distributions maximum and minimum values. The simulations for the two-way classification procedure produced several occurrences within each hypothesised model of a negative interaction component. The frequency and magnitude of these negative components increased as the size of the concentration parameter, k, decreased. When simulations were run to further examine the power of the tests and specified differences were given between treatments, blocks or cells, (i.e. separate von Mises distributions with equal concentration parameters but differing mean directions) the frequency of the negative interaction components increased.

An example of the two-way classification with interaction and its analysis is given in Tables 7.2.1 and 7.2.2. It demonstrates the situation where the individual factor concentration parameters (i.e. k_1, k_2,.....) are not significantly different and are large, but the overall concentration parameter k is small. Other less dramatic data sets may be shown with a larger overall concentration parameter but producing similar, although less pronounced, results.

Example 7.2.1

Table 7.2.1 A Two-way Classification

		FACTOR		
		A_0	A_1	
FACTOR	B_0	$61^{\bullet}$ $48^{\bullet}$ $87^{\bullet}$ $R_{11.} = 4.7303$ $102^{\bullet}$ $N_{11.} = 5$ $73^{\bullet}$	$177^{\bullet}$ $139^{\bullet}$ $155^{\bullet}$ $R_{12.} = 4.7604$ $191^{\bullet}$ $N_{12.} = 5$ $164^{\bullet}$	$R_{1..} = 6.65$ $N_{1.} = 10$
	B_1	$318^{\bullet}$ $353^{\bullet}$ $358^{\bullet}$ $R_{21.} = 4.8284$ $328^{\bullet}$ $N_{21.} = 5$ $344^{\bullet}$	$240^{\bullet}$ $281^{\bullet}$ $229^{\bullet}$ $R_{22.} = 4.7623$ $252^{\bullet}$ $N_{22.} = 5$ $240^{\bullet}$	$R_{2..} = 6.6548$ $N_{2..} = 10$
		$R_{.1.} = 6.5247$ $N_{.1.} = 10$	$R_{.2.} = 7.1363$ $N_{.2.} = 10$	$R_{...} = 0.6117$ $N = 20$

$$\sum_{i=1}^{p} R_{i..} = 13.3058 \qquad \sum_{j=1}^{q} R_{.j.} = 13.661 \qquad \sum_{i=1}^{p} \sum_{j=1}^{q} R_{ij.} = 19.0814$$

$p = q = 2, \ l = 5$

Table 7.2.2 Two-Way Analysis of Variance Table

Variation Component		Value	Degrees of Freedom
Due to Factor A	$\sum_{i=1}^{p} R_{i..} - R_{...}$	12.6931	$(p-1) = 1$
Due to Factor B	$\sum_{j=1}^{q} R_{.j.} - R_{...}$	13.0493	$(q-1) = 1$
Inter-action	$\sum_{i=1}^{p} \sum_{j=1}^{q} R_{ij.} - \sum_{i=1}^{p} R_{i..} - \sum_{j=1}^{q} R_{.j.} + R_{...}$	-7.2727	$(p-1)(q-1)=1$
Residual	$N - \sum_{i=1}^{p} \sum_{j=1}^{q} R_{ij.}$	0.9186	$pq(l-1) = 16$
Total	$N - R_{...}$	19.3883	$N - 1 = 19$

It should be noted from Table 7.2.2 that the sum of the two main effects measure of variation is greater than the total measure of variation and therefore the interaction measure is negative.

As noted the above example does not satisfy the assumption of large overall concentration parameter k but has been used to illustrate the associated problems when k approaches and decreases below 2. This type of data set questions the independence of the test components in less dramatic cases. [The presence of a negative interaction measure of variation does not occur if the data is axial in nature and the general assumptions are upheld.]

7.3 A Further Approach to Watson and Williams Design Using a Regression Model

Section 7.2 has given an example of where the extended techniques for the randomised and two-way analysis designs may breakdown. In this section we will briefly reiterate how the one-way analysis design was constructed and then investigate, via a regression approach, the structure of the individual components. From this approach the possible reasons for the breakdown of the present statistics in larger designs may then be seen.

When testing the hypothesis that there is no difference between a number of treatments, a basic result given in all texts, for the simple linear one-way analysis, is

$$\sum_{i=1}^{p} \sum_{j=1}^{q} (x_{ij} - \bar{x})^2 = q \sum_{i=1}^{p} (\bar{x}_i - \bar{x})^2 + \sum_{i=1}^{p} \sum_{j=1}^{q} (x_{ij} - \bar{x}_i)^2 \qquad (7.3.1)$$

where x_{ij} represents the individual observations, $\bar{x}_i$ the treatment sample mean values and $\bar{x}$ the overall sample mean. The left hand side measures the dispersion or scatter of the whole sample about the true mean. The first term on the right hand side measures the dispersion of the sample about the estimated mean $\bar{x}$, whilst the last term measures the dispersion of the sample means about the true mean.

It is easily shown that $N-R_{..}$ represents the dispersion of the sample of directions about the <u>estimated</u> mean direction in circular statistics. Equally $N-X$ represents the dispersion of the sample about the <u>true</u> mean direction, giving the expression

$$(N - X) = (N - R_{..}) + (R_{..} - X) \qquad (7.3.2)$$

in our analogue of (7.3.1).

Using the results of (4.2.3)

$$2k(N - X) = 2k(N - R_{..}) + 2k(R_{..} - X) \tag{7.3.3}$$

has the associated chi-squared distributions

$$\chi^2_N = \chi^2_{N-1} + \chi^2_1$$

This was developed further by Watson (1956) to examine 2 or more samples with assumed equal concentration parameter. It may be seen that if the mean direction of two samples differ greatly the sum of their resultants $R_{.1}$ and $R_{.2}$, would be much greater than the overall resultant $R_{..}$. Similarly, if the mean directions are equal $R_{.1} + R_{.2} = R_{..}$. This is then used as a measure of variation between samples. The variation within the samples is measured by comparing the maximum length of a sample resultant, $N_{.j}$, to its actual resultant, $R_{.j}$. For a two sample test the within variation is therefore

$$(N_{.1} - R_{.1}) + (N_{.2} - R_{.2})$$

This suggests the analysis of variance expression

$$2k(N - R_{..}) = 2k(N_{.1} - R_{.2} + N_{.2} - R_{.2}) + 2k(R_{.1} + R_{.2} - R_{..}) \tag{7.3.4}$$

It is easy to see from these analogies how the test for the one way analysis is derived. Nevertheless, it is not until we investigate the individual components of the analysis of variance expression that we see an underlying problem that is not fully appreciated until larger, more complex designs are constructed.

A simple alternative way of building the analysis of variance expression is via a regression approach using basic vector analysis. Figure 7.3.1 gives the notation that we will use to construct the components.

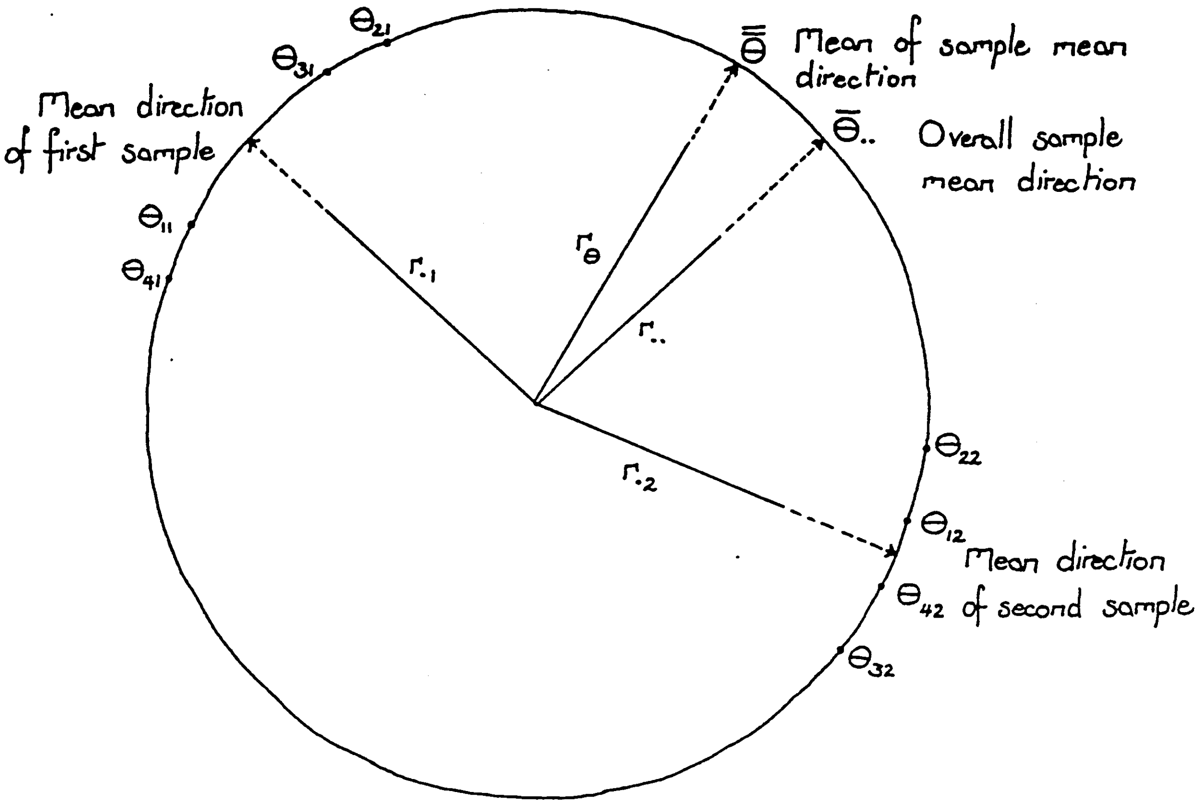


Figure 7.3.1 Angular Notation for the Construction of Components

Assuming the θ_{ij}'s (i=1,2,...,p; j=1,2,...,q) are independently distributed as $M(\mu_0+\beta_j,k)$, where μ_0 is some mean direction and β_j is the possible effect due to treatment j. Let us test

$$H_0 : \beta_1 = \beta_2 = \ldots\ldots\ldots = \beta_q = 0$$

$$H_1 : \text{at least one } \beta_j \neq 0$$

Let $\hat{\mu}_0$ and $(\hat{\mu}_0, \hat{\beta}_j)$ be the maximum likelihood estimates of μ_0 and $\mu_0; \beta_j$ under H_0 and H_1. Let

$$S_0 = N - \sum_{i=1}^{p} \sum_{j=1}^{q} \cos(\theta_{ij} - \hat{\mu}_0 - \hat{\beta}_j)$$

From the statistical results of (4.2.3), $2kS_0$ has a χ^2 distribution with N degrees of freedom, for large k. Using approximation (1.4.13), $2kS_0$ is equivalent to

$$k \sum_{i=1}^{p} \sum_{j=1}^{q} (\theta_{ij} - \hat{\mu}_0 - \hat{\beta}_j)^2 \tag{7.3.6}$$

Using this approximation we may construct the individual components of the expression.

7.3.1 Total Measure of Variation

From equation (7.3.6) the component for the total measure of variation is calculated from the sum of the squared distances between each sample point, θ_{ij} and the overall sample mean direction, $\bar{\theta}_{..}$, as illustrated in Figure 7.3.2.

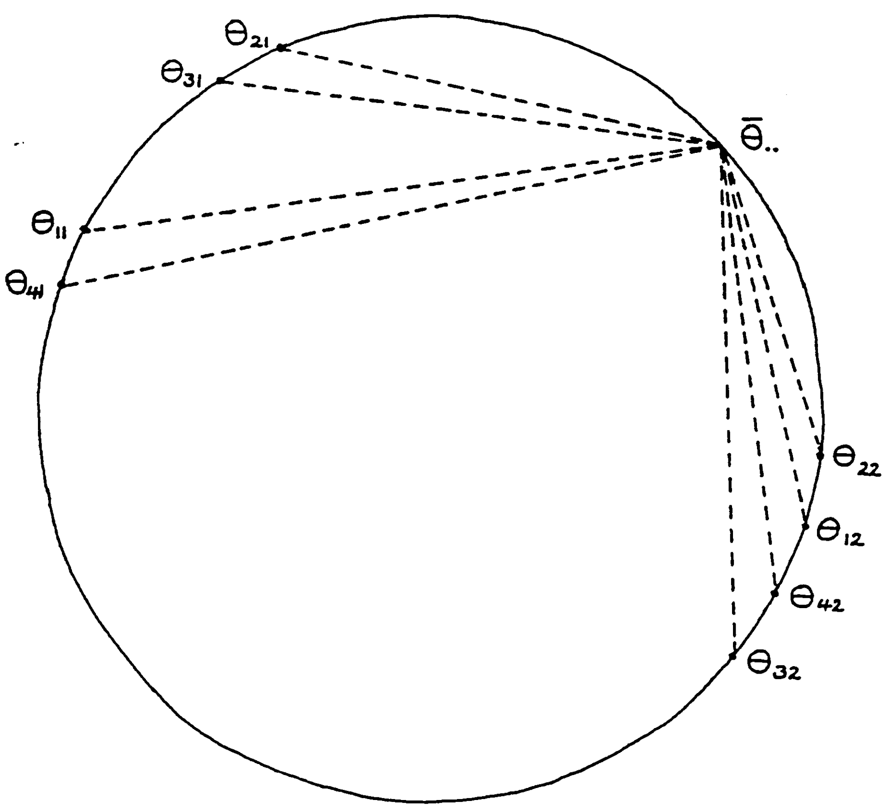


Figure 7.3.2 Distances Between the Sample Points, θ_{ij} and the Overall Sample Mean Direction, $\bar{\theta}_{..}$

From vector algebra and basic directional data properties we have, denoting $|\theta_{ij}|$ as the vector length;

$$k\sum_{i=1}^{p}\sum_{j=1}^{q}(\underline{\theta}_{ij} - \underline{\bar{\theta}}_{..})^2 \tag{7.3.7}$$

$$= k\sum_{i=1}^{p}\sum_{j=1}^{q}(|\underline{\theta}_{ij}|^2 + |\underline{\bar{\theta}}_{..}|^2 - 2\cos(\theta_{ij} - \bar{\theta}_{..}))$$

$$= k(N + N - 2R_{..})$$

$$= 2k(N - R_{..}) \tag{7.3.8}$$

Producing the same total measure of variation as in (7.3.4).

7.3.2 Residual or Within Measure of Variation

The residual measure of variation is calculated from the sum of the squared distances between each sample point, θ_{ij}, and its own sample mean direction, $\bar{\theta}_{.j}$

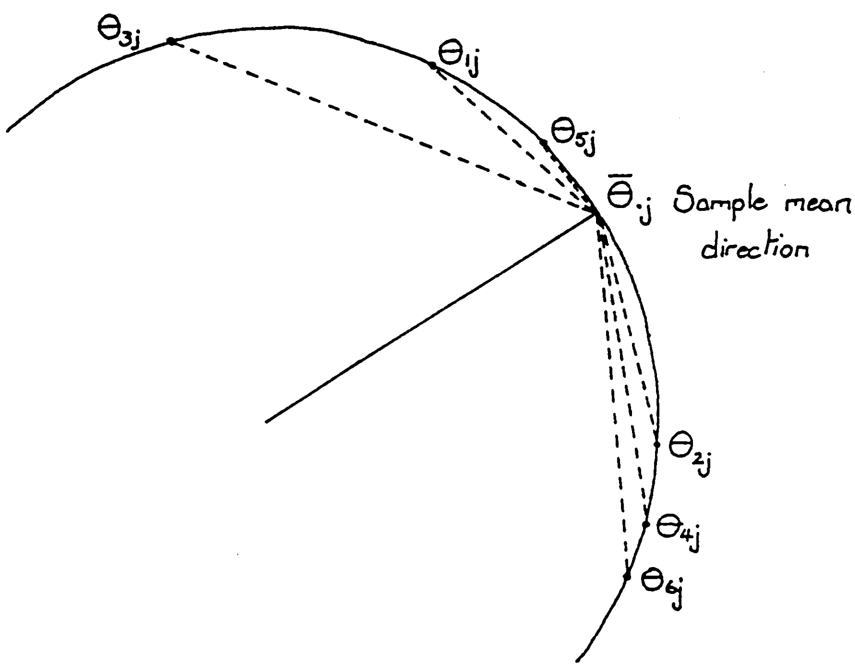


Figure 7.3.3 Distances Between Sample Points, θ_{ij} and Sample Mean Directions, $\bar{\theta}_{.j}$

$$k\sum_{i=1}^{p}\sum_{j=1}^{q}(\underline{\theta}_{ij} - \overline{\underline{\theta}}_{.j})^2 \qquad (7.3.9)$$

$$= k\sum_{i=1}^{p}\sum_{j=1}^{q}(|\underline{\theta}_{ij}|^2 + |\overline{\underline{\theta}}_{.j}|^2 - 2\cos(\theta_{ij} - \overline{\theta}_{.j}))$$

$$= k(N + N - 2\sum_{j=1}^{q} R_{.j})$$

$$= 2k(N - \sum_{j=1}^{q} R_{.j}) \qquad (7.3.10)$$

Producing the same residual measure as in (7.3.4).

7.3.3 Between Measure of Variation

If, as is shown in 'linear' statistical analysis, the total measure of variation is split into two parts, a residual or within measure and a between measure, then the between measure may be found from

$$2k(N - R_{..}) - 2k(N - \sum_{j=1}^{q} R_{.j}) = 2k(\sum_{j=1}^{q} R_{.j} - R_{..}) \qquad (7.3.11)$$

However, equation (7.3.11) assumes, as in 'linear' analysis, that when circular mean directions are combined the same overall mean direction is produced. It was first observed in Example 5.3.1 that this property does not exist for circular statistical analysis, the following sub-section investigates this prior to deriving the 'true' between measure of variation.

7.3.3.1 Combining Mean Angular Directions and Resultant Lengths

Expressed algebraically by (7.3.12) a simple proof shows that when angular mean directions are combined the overall angular mean direction may not be produced, and is dependent on the resultant lengths found within each of the combined samples.

$$\frac{\sum_{j=1}^{q} \sin \bar{\theta}_{.j}}{\sum_{j=1}^{q} \cos \bar{\theta}_{.j}} \neq \frac{\sum_{i=1}^{p} \sum_{j=1}^{q} \sin \theta_{ij}}{\sum_{i=1}^{p} \sum_{j=1}^{q} \cos \theta_{ij}} \qquad (7.3.12)$$

An angular mean direction, $\bar{\theta}_{..}$, is given by

$$\cos \bar{\theta}_{..} = \frac{\bar{C}}{\bar{R}} \qquad \sin \bar{\theta}_{..} = \frac{\bar{S}}{\bar{R}} \qquad (7.3.13)$$

Similarly for $\bar{\theta}_{.j}$

$$\cos \bar{\theta}_{.j} = \frac{\bar{C}_{.j}}{\bar{R}_{.j}} \qquad \sin \bar{\theta}_{.j} = \frac{\bar{S}_{.j}}{\bar{R}_{.j}}$$

Therefore

$$\frac{\sum_{j=1}^{q} \sin \bar{\theta}_{.j}}{\sum_{j=1}^{q} \cos \bar{\theta}_{.j}} = \frac{\sum_{j=1}^{q} \frac{1}{\bar{R}_{.j}} \sum_{i=1}^{p} \sin \theta_{ij}}{\sum_{j=1}^{q} \frac{1}{\bar{R}_{.j}} \sum_{i=1}^{p} \cos \theta_{ij}} \neq \frac{\sum_{i=1}^{p} \sum_{j=1}^{q} \sin \theta_{ij}}{\sum_{i=1}^{p} \sum_{j=1}^{q} \cos \theta_{ij}} \qquad (7.3.14)$$

Figure 7.3.1 gave the mean direction of the sample means as $\bar{\bar{\theta}}$ with resultant length R_{θ}. This result gives further understanding and an alternative calculation of the overall mean direction and its associated resultant length. By utilising the resultant lengths associated with each sample mean direction, the mean directions may be combined, as in standard statistical analysis, to give the overall mean direction. Using a rectangular co-ordinate system with X and Y axes and origin 0, let $\bar{\theta}_{.j}$ be one of the q mean angles with corresponding mean resultant length $r_{.j}$. Let $x_{.j}$ and $y_{.j}$ be the rectangular components of $r_{.j}$, as seen in Figure 7.3.4. Then by definition

$$x_{.j} = r_{.j}\cos\bar{\theta}_{.j} \qquad y_{.j} = r_{.j}\sin\bar{\theta}_{.j} \tag{7.3.15}$$

Then

$$x_{..} = \frac{1}{q}(r_{.1}\cos\bar{\theta}_{.1} + r_{.2}\cos\bar{\theta}_{.2} + \ldots\ldots + r_{.q}\cos\bar{\theta}_{.q}) \tag{7.3.16}$$

$$y_{..} = \frac{1}{q}(r_{.1}\sin\bar{\theta}_{.1} + r_{.2}\sin\bar{\theta}_{.2} + \ldots\ldots + r_{.q}\sin\bar{\theta}_{.q}) \tag{7.3.17}$$

Therefore

$$r_{..} = \left[\left(\frac{1}{q}\sum_{j=1}^{q} r_{.j}\cos\bar{\theta}_{.j}\right)^2 + \left(\frac{1}{q}\sum_{j=1}^{q} r_{.j}\sin\bar{\theta}_{.j}\right)^2\right]^{\frac{1}{2}} \tag{7.3.18}$$

$$R_{..} = Nr_{..}$$

Similarly for p row mean directions

$$r_{..} = \left[\left(\frac{1}{p}\sum_{i=1}^{p} r_{i.}\cos\bar{\theta}_{i.}\right)^2 + \left(\frac{1}{p}\sum_{i=1}^{p} r_{i.}\sin\bar{\theta}_{i.}\right)^2\right]^{\frac{1}{2}} \tag{7.3.19}$$

Giving the overall mean direction as

$$\bar{\theta}_{..} = \text{arc tan}\left[\frac{\frac{1}{q}\sum_{j=1}^{q} r_{.j}\sin\bar{\theta}_{.j}}{\frac{1}{q}\sum_{j=1}^{q} r_{.j}\cos\bar{\theta}_{.j}}\right] \tag{7.3.20}$$

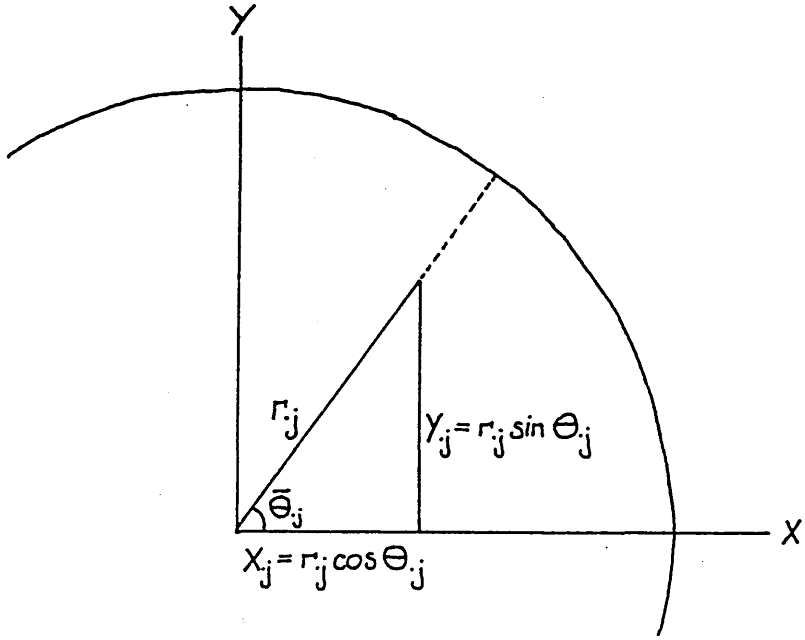


Figure 7.3.4 The Rectangular Components of a Non-Unit Vector

This may be further understood by a simple illustration given in Figure 7.3.5. Here two samples have four observations within each, by taking account of the sample mean resultant lengths the true overall mean direction is obtained.

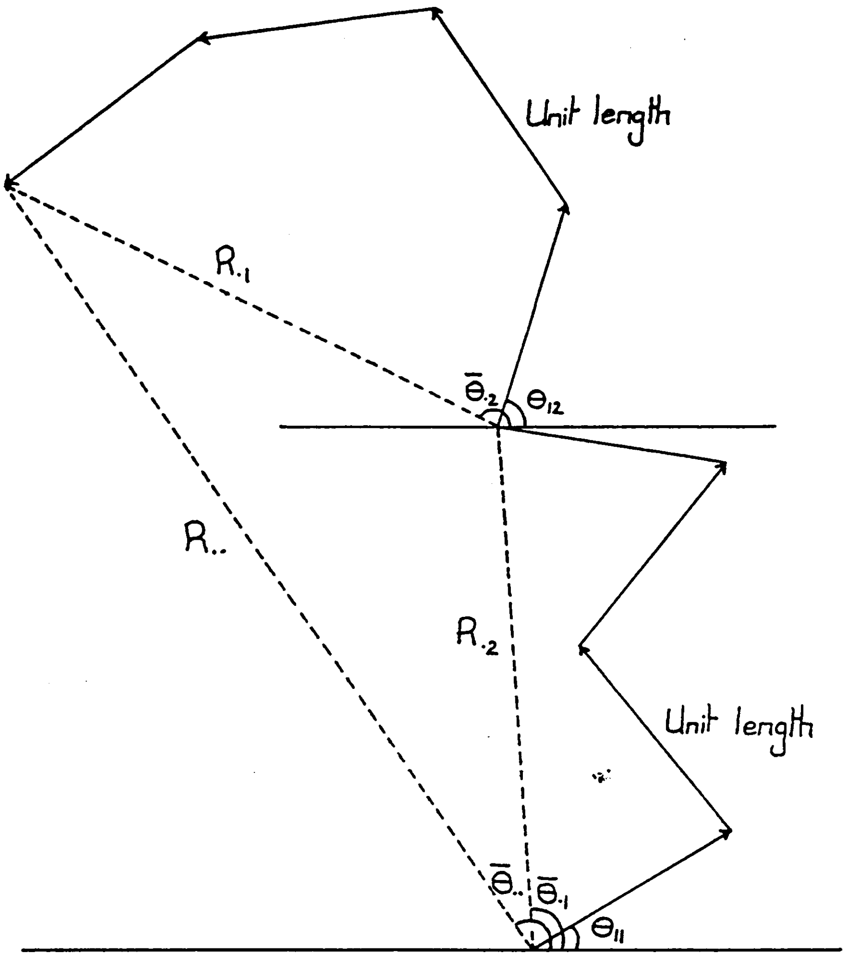


Figure 7.3.5 Combining Samples Using Angular Mean Directions and their Mean Resultant Lengths

Note: (a) Fisher and Lewis (1983) considered the problem of forming a pooled estimate of the common mean direction of several circular samples with possibly differing concentration parameters. In brief their work discussed the introduction of some arbitrary non-random weighting factor to each sample defining

$$\bar{C}_w = \sum_{i=1}^{q} w_i \, \bar{C}_i \qquad \bar{S}_w = \sum_{i=1}^{q} w_i \, \bar{S}_i$$

$$\bar{R}_w^2 = \bar{C}_w^2 + \bar{S}_w^2 \qquad\qquad (w_i > 0 \sum_{i=1}^{q} w_i = 1)$$

The general pooled estimate for μ, given by $\hat{\mu}_w$, defined by

$$\cos\hat{\mu}_W = \frac{\bar{C}_W}{\bar{R}_W} \qquad \sin\hat{\mu}_W = \frac{\bar{S}_W}{\bar{R}_W}$$

Using the central limit theorem Fisher and Lewis produced a confidence interval of the pooled estimate comparing this to the approximate confidence cone for the single sample case. (Further approximate confidence intervals for a mean direction of a von Mises distribution were later given by Upton (1986)). Fisher and Lewis showed the consequences of specific choices of weights, considering equal weighting of 1/q and proportional weighting of N_i/N.

(b) An alternative calculation of the overall resultant length from the sample angular mean directions and resultant lengths is via the dot product rule for vectors

Using the q column statistics

$$R^2_{..} = \sum_{j=1}^{q} R^2_{.j} + 2\sum_{\substack{j=1 \\ j \neq t \\ j < t}}^{q} \sum_{t=1}^{q} R_{.j}\, R_{.t} \cos(\bar{\theta}_{.j} - \bar{\theta}_{.t}) \tag{7.3.21}$$

Similarly for the p row statistics

$$R^2_{..} = \sum_{i=1}^{p} R^2_{i.} + 2\sum_{\substack{i=1 \\ i \neq t \\ i < t}}^{p} \sum_{t=1}^{p} R_{i.}\, R_{t.} \cos(\bar{\theta}_{i.} - \bar{\theta}_{t.}) \tag{7.3.22}$$

For example, if the overall resultant was to be found from three sets of sample statistics

$$\begin{aligned} R^2_{..} = R^2_{1.} + R^2_{2.} + R^2_{3.} &+ 2R_{1.}R_{2.}\cos(\bar{\theta}_{1.} - \bar{\theta}_{2.}) \\ &+ 2R_{1.}R_{3.}\cos(\bar{\theta}_{1.} - \bar{\theta}_{3.}) \\ &+ 2R_{2.}R_{3.}\cos(\bar{\theta}_{2.} - \bar{\theta}_{3.}) \end{aligned}$$

Returning to the calculation of the between measure of variation, it is evident that this should be calculated from the sum of squared distances between each sample mean direction $\bar{\theta}_{.j}$ and their combined mean direction $\bar{\bar{\theta}}$. As an identity this will leave a further component measuring the difference between the combined mean direction, $\bar{\bar{\theta}}$, and the overall mean direction, $\bar{\theta}_{..}$, as in (7.3.23)

$$(\theta_{ij} - \bar{\theta}_{..}) = (\theta_{ij} - \bar{\theta}_{.j}) + (\bar{\theta}_{.j} - \bar{\bar{\theta}}) + (\bar{\bar{\theta}} - \bar{\theta}_{..}) \qquad (7.3.23)$$

total	residual	between	difference between the combined mean direction and the overall mean direction.

Therefore the 'true' between measure of variation is given by;

$$k \sum_{i=1}^{p} \sum_{j=1}^{q} (\bar{\underline{\theta}}_{.j} - \bar{\bar{\underline{\theta}}})^2$$

$$= k \sum_{i=1}^{p} \sum_{j=1}^{q} (|\bar{\underline{\theta}}_{.j}|^2 + |\bar{\bar{\underline{\theta}}}|^2 - 2\cos(\bar{\theta}_{.j} - \bar{\bar{\theta}}))$$

$$= k(N + N - 2R_{\theta})$$

$$= 2k(N - R_{\theta}) \qquad (7.3.24)$$

7.4 Cross Product Terms and the Analysis of Cross-Classification

In 'linear' statistics the corresponding expression to (7.3.23) will produce cross product terms equal to zero. On the circle these terms are found to be non-zero;

$$\sum_{i=1}^{p} \sum_{j=1}^{q} (\underline{\theta}_{ij} - \bar{\underline{\theta}}_{..})^2 = \sum_{i=1}^{p} \sum_{j=1}^{q} (\underline{\theta}_{ij} - \bar{\underline{\theta}}_{.j})^2 + \sum_{i=1}^{p} \sum_{j=1}^{q} (\bar{\underline{\theta}}_{.j} - \bar{\bar{\underline{\theta}}})^2$$

$$+ \sum_{i=1}^{p} \sum_{j=1}^{q} (\bar{\bar{\underline{\theta}}} - \bar{\underline{\theta}}_{..})^2 + 2 \sum_{i=1}^{p} \sum_{j=1}^{q} (\underline{\theta}_{ij} - \bar{\underline{\theta}}_{.j})(\bar{\underline{\theta}}_{.j} - \bar{\bar{\underline{\theta}}})$$

$$+ 2\sum_{i=1}^{p}\sum_{j=1}^{q}(\bar{\underline{\theta}}_{.j} - \bar{\bar{\underline{\theta}}})(\bar{\bar{\underline{\theta}}} - \bar{\underline{\theta}}_{..}) + 2\sum_{i=1}^{p}\sum_{j=1}^{q}(\underline{\theta}_{ij} - \bar{\underline{\theta}}_{.j})(\bar{\bar{\underline{\theta}}} - \bar{\underline{\theta}}_{..}) \tag{7.4.1}$$

The left hand side and the first two terms on the right hand side of (7.4.1) have been simplified in Section 7.3, giving

$$k\sum_{i=1}^{p}\sum_{j=1}^{q}(\underline{\theta}_{ij} - \bar{\underline{\theta}}_{..})^2 = 2k(N - R_{..}) \qquad \text{from (7.3.8)}$$

$$k\sum_{i=1}^{p}\sum_{j=1}^{q}(\underline{\theta}_{ij} - \bar{\underline{\theta}}_{.j})^2 = 2k(N - \sum_{j=1}^{q} R_{.j}) \qquad \text{from (7.3.10)}$$

$$k\sum_{i=1}^{p}\sum_{j=1}^{q}(\bar{\underline{\theta}}_{.j} - \bar{\bar{\underline{\theta}}})^2 = 2k(N - R_{\theta}) \qquad \text{from (7.4.24)}$$

The following term is the measure of the difference between the combined means and the overall mean direction.

$$k\sum_{i=1}^{p}\sum_{j=1}^{q}(\bar{\bar{\underline{\theta}}} - \bar{\underline{\theta}}_{..})^2 = kN(|\bar{\bar{\underline{\theta}}}|^2 + |\bar{\underline{\theta}}_{..}|^2 - 2\cos(\bar{\bar{\theta}} - \bar{\theta}_{..})$$

$$= 2kN(1 - \cos(\bar{\bar{\theta}} - \bar{\theta}_{..})) \tag{7.4.2}$$

The final three terms are the corresponding cross product terms.

$$2k\sum_{i=1}^{p}\sum_{j=1}^{q}(\underline{\theta}_{ij} - \bar{\underline{\theta}}_{.j})(\bar{\underline{\theta}}_{.j} - \bar{\bar{\underline{\theta}}}) = 2kp\sum_{j=1}^{q}(\frac{R_{.j}}{p}\bar{\underline{\theta}}_{.j})(\bar{\underline{\theta}}_{.j} - \bar{\bar{\underline{\theta}}})$$

$$2k\sum_{i=1}^{p}\sum_{j=1}^{q}(\bar{\underline{\theta}}_{.j} - \bar{\bar{\underline{\theta}}})(\bar{\bar{\underline{\theta}}} - \bar{\underline{\theta}}_{..}) = 2kN(\frac{R_{\theta}}{N} - \bar{\bar{\underline{\theta}}})(\bar{\bar{\underline{\theta}}} - \bar{\underline{\theta}}_{..})$$

$$2k\sum_{i=1}^{p}\sum_{j=1}^{q}(\underline{\theta}_{ij} - \underline{\bar{\theta}}_{.j})(\underline{\bar{\bar{\theta}}} - \underline{\bar{\theta}}_{..}) = 2kN(\frac{R_{..}}{N} - \frac{R_\theta}{N})(\underline{\bar{\bar{\theta}}} - \underline{\bar{\theta}}_{..}) \qquad (7.4.3)$$

Summing the cross product terms of (7.4.3) equals

$$= 2k(\sum_{j=1}^{q} R_{.j} - 2N + R_\theta - R_{..} + N\cos(\bar{\bar{\theta}} - \bar{\theta}_{..})) \qquad (7.4.4)$$

Producing the circular model

$$2k(N - R_{..}) = 2k(N - \sum_{j=1}^{q} R_{.j}) + 2k(N - R_\theta) + 2kN(1 - \cos(\bar{\bar{\theta}} - \bar{\theta}_{..}))$$

(Total) (Residual) (Between) (Difference between combined and overall mean.)

$$+ 2k(N - \sum_{j=1}^{q} R_{.j} - 2N + R_\theta - R_{..} + N\cos(\bar{\bar{\theta}} - \bar{\theta}_{..}))$$

(Cross product terms)

Reducing

$$2k(N - R_{..}) = 2k(N - \sum_{j=1}^{q} R_{.j}) + 2k(N - R_\theta)$$

(Total) (Residual) (Between)

$$+ 2k(\sum_{j=1}^{q} R_{.j} + R_\theta - R_{..} - N) \qquad (7.4.5)$$

(Correction)

Unless the residual term does not exist the correction will always be negative since the between measure in (7.4.5) will always be greater than the between measure in (7.3.4) from Watson and Williams.

When larger designs are considered in 'linear' statistics the between sum of squares can be broken down into three components measuring differences between rows, differences between columns and the interaction within the design. In directional analysis, from extending Watson and Williams, this splitting produces:-

$$\left(\sum_{i=1}^{p}\sum_{j=1}^{q} R_{ij.} - R_{...}\right) = \underset{\text{(Between rows)}}{\left(\sum_{i=1}^{p} R_{i..} - R_{...}\right)} + \underset{\text{(Between columns)}}{\left(\sum_{j=1}^{q} R_{.j.} - R_{...}\right)}$$

$$+ \underset{\text{(Interaction)}}{\left(\sum_{i=1}^{p}\sum_{j=1}^{q} R_{ij.} - \sum_{i=1}^{p} R_{i..} - \sum_{j=1}^{q} R_{.j.} + R_{...}\right)} \qquad (7.4.6)$$

If the first and second terms on the right hand side of (7.4.6) are calculated their sum may be greater than the total measure of variation, as was illustrated in Example 5.3.1. In this situation the value of the interaction will be negative, whether interaction exists or not. It is only when the between measure is broken down in this manner for larger designs is the problem within its derivation fully realised.

7.5 Summary

Chapter 7 has shown via a simple alternative approach to Watson and Williams how,as k decreases, the model components begin to breakdown. The most obvious consequence is seen when analysing two-way classification designs when a negative component may be produced. These are shown to be a consequence of the way in which circular mean directions combine. Unlike linear statistics, if the overall mean direction of a sample is found, $\bar{\theta}_{..}$, and then the same sample is split into equally weighted samples and the overall mean direction re-calculated from the resulting combined means, a different overall direction, $\bar{\bar{\theta}}$, may be produced.

For the one-way analysis this does not cause any major problems. However, when the technique is extended to larger analyses the sum of the main effect components may give a result greater than the total measure of variation.

It is important to re-emphasise that the above only occurs as the overall concentration parameter k decreases towards and below 2. The new extended procedures, developed in Chapter 6, are satisfactory for highly clustered data sets since the larger the value of the overall concentration parameter the smaller the correction value of Section 7.4, and similarly the closer the two mean directions, $\bar{\theta}_{..}$ and $\bar{\bar{\theta}}$, become.

CHAPTER 8

DEVELOPMENT OF A NEW ANALYSIS OF VARIANCE PROCEDURE

8.1 Introduction

Chapters 6 and 7 have shown (1) how the modified components for the one–way classification have excellent chi–squared approximations for large k (2) how the extended components for further design models also follow good chi–squared approximations, for large k (3) how the underlying structure of the components may breakdown as k decreases (4) the dependence of the components as k varies and (5) since the overall concentration parameter must be large as well as the sample concentration parameters, the tests may be good approximations only when applied to highly clustered data sets.

This chapter is concerned with developing new test statistics which may be generalised across all values of k and are based on likelihood ratio statistics for testing both the mean direction and the concentration parameter, k.

The main reason for the breakdown of the extended models as k diminishes is concerned with the combining of the angular mean directions, discussed in Section 7.3. A new procedure is shown to overcome this problem and therefore enable components such as interaction to be investigated as k decreases.

Section 7.33 showed how the overall mean direction will remain unaltered if the sample resultant lengths are retained when combining sample mean directions. Using this fact, Section 8.2 constructs new components via a regression approximation approach previously utilised to show the construction of Watson and Williams original test statistic.

Following the discussion of the cross product terms in Chapter 7, Section 8.4 discusses the interpretation and calculation of interaction for the procedures.

8.2 Minimising Chord Distances to Build New Test Statistic Components

Using exactly the same approach as Section 7.3 a design procedure may be built via a regression approach using basic vector analysis but also taking account of the mean resultant lengths. This method will minimise the chord lengths between points within the circle.

Assume the θ_{ij}'s ($i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, q$) are independently distributed as $M(\mu_0 + \beta_j, k)$ where μ_0 is some overall mean direction and β_j is the possible effect due to treatment j. The null and alternative hypotheses are

$$H_0 : \beta_1 = \beta_2 = \ldots\ldots = \beta_q \qquad k_1 = k_2 = \ldots\ldots = k_q$$

against

$$H_1 : \beta_1 \neq \beta_2 \neq \ldots\ldots \neq \beta_q \qquad k_1 \neq k_2 \neq \ldots\ldots \neq k_q$$

or the testing of different populations. To overcome this, the equality of the concentration parameters must be examined prior to any examination of the main effects, in a similar manner to that undertaken in standard 'linear' analysis of variance.

8.2.1 Total Measure of Variation (TMV)

It was shown in Section 7.3 that $2kS_0$ is equivalent to

$$k \sum_{i=1}^{p} \sum_{j=1}^{q} (\theta_{ij} - \hat{\mu}_0 - \hat{\beta}_j)^2 \qquad (8.2.1)$$

Hence under H_0:

$$k\sum_{i=1}^{p}\sum_{j=1}^{q}(\theta_{ij}-\mu_0-\beta_j)^2 = k\sum_{i=1}^{p}\sum_{j=1}^{q}(\theta_{ij}-\bar{\theta}_{..})^2$$

Using the above expression the total measure of variation will be calculated from the sum of the squared distances between each sample point, θ_{ij}, and the overall sample mean direction, $\bar{\theta}_{..}$, as illustrated in Figure 8.2.1.

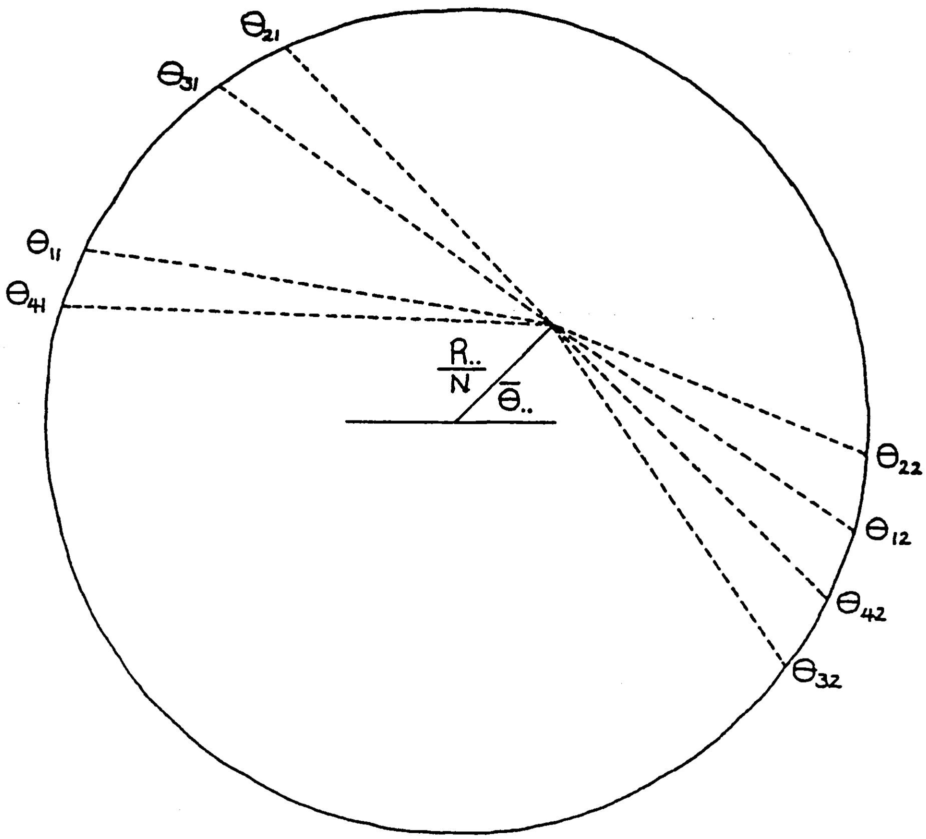


Figure 8.2.1 Vector Lengths for the Total Measure of Variation

Here the mean resultant length of $\bar{\theta}_{..}$ is utilised rather than effectively extending it to the circumference of the circle. From vector algebra and basic directional data properties we have, denoting $|\theta_{ij}|$ as the vector length:

$$k\sum_{i=1}^{p}\sum_{j=1}^{q}(\underline{\theta}_{ij} - \underline{\bar{\theta}}_{..})^2 \tag{8.2.2}$$

$$= k\sum_{i=1}^{p}\sum_{j=1}^{q}\left[|\underline{\theta}_{ij}|^2 + |\underline{\bar{\theta}}_{..}|^2 - 2|\underline{\theta}_{ij}||\underline{\bar{\theta}}_{..}|\cos(\theta_{ij} - \bar{\theta}_{..})\right]$$

$$= k\sum_{i=1}^{p}\sum_{j=1}^{q}\left[1 + \frac{R^2_{..}}{N^2_{..}} - \frac{2R_{..}}{N_{..}}\cos(\theta_{ij} - \bar{\theta}_{..})\right]$$

$$= k\left[N + \frac{R^2_{..}}{N_{..}} - \frac{2R^2_{..}}{N_{..}}\right]$$

$$\text{TMV} = k\left[N - \frac{R^2_{..}}{N_{..}}\right] \tag{8.2.3}$$

8.2.2 Residual Measure of Variation (RMV)

The residual measure of variation will be calculated from the sum of the squared distances between each sample point, θ_{ij}, and its own sample mean direction, $\bar{\theta}_{.j}$, as illustrated in Figure 8.2.2.

$$k\sum_{i=1}^{p}\sum_{j=1}^{q}(\underline{\theta}_{ij} - \underline{\bar{\theta}}_{.j})^2 \tag{8.2.4}$$

$$= k\sum_{i=1}^{p}\sum_{j=1}^{q}\left[|\underline{\theta}_{ij}|^2 + |\underline{\bar{\theta}}_{.j}|^2 - 2\,|\underline{\theta}_{ij}||\underline{\bar{\theta}}_{.j}|\cos(\theta_{ij} - \bar{\theta}_{.j})\right]$$

$$= k\sum_{i=1}^{p}\sum_{j=1}^{q}\left[1 + \frac{R^2_{.j}}{N^2_{.j}} - 2\left[\frac{R_{.j}}{N_{.j}}\right]\cos(\theta_{ij} - \bar{\theta}_{.j})\right]$$

$$= k\left[N + \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - 2\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right]\right]$$

$$RMV = k\left[N - \sum_{j=1}^{q}\left[\frac{R^2_{\cdot j}}{N_{\cdot j}}\right]\right] \tag{8.2.5}$$

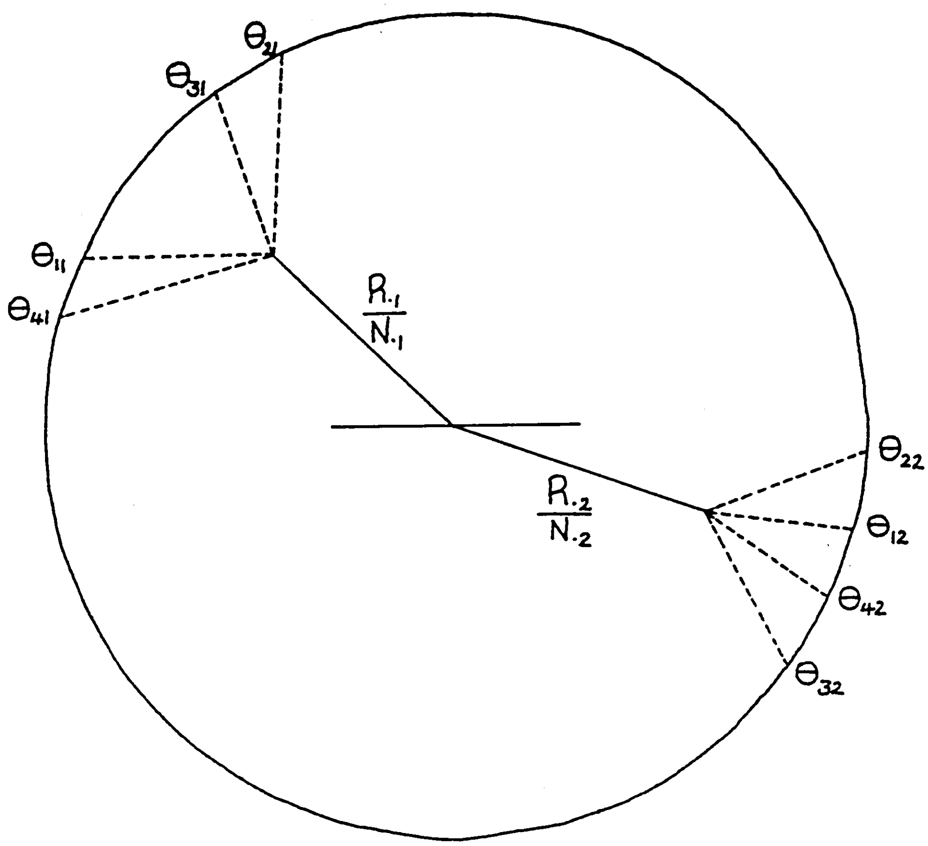


Figure 8.2.2 Vector Lengths for the Residual Measure of Variation

8.2.3 Between Measure of Variation (BMV)

By incorporating the resultant length with its respective angular mean, as discussed in Section 7.3.3, we may produce the same overall resultant length and angular mean when they are combined.

Let us now construct the between measure of variation in the same manner as for the residual and total measures. The between component will be calculated from the sum of the squared distances between the sample angular means and the overall angular mean. This measure is illustrated in Figure 8.2.3.

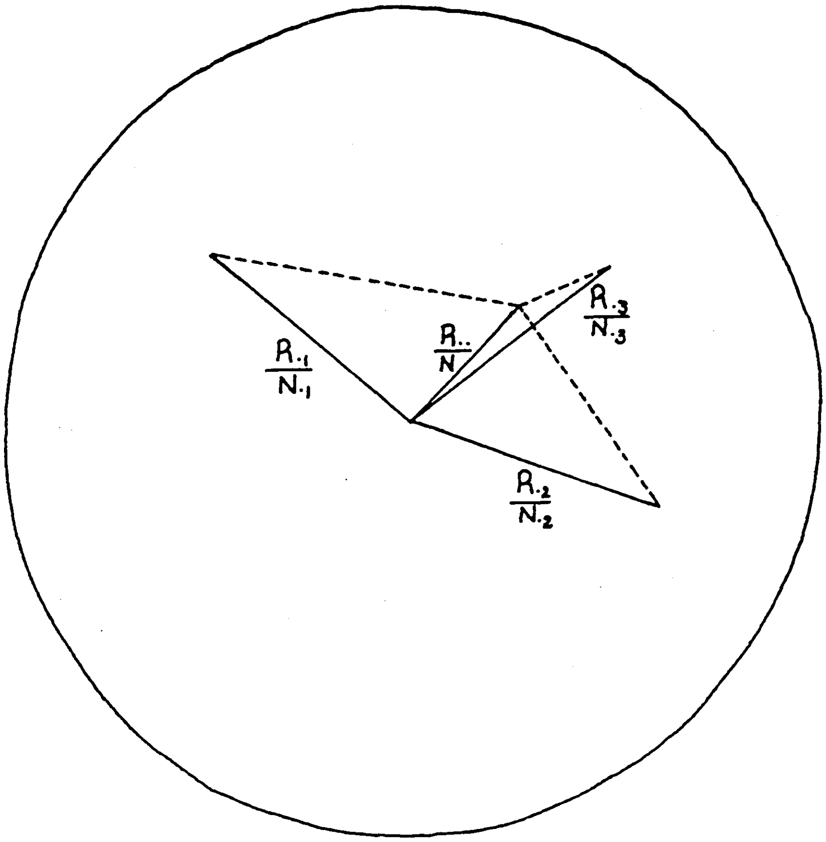


Figure 8.2.3 Vector Length for the Between Measure of Variation

$$k \sum_{i=1}^{p} \sum_{j=1}^{q} (\underline{\bar{\theta}}_{.j} - \underline{\bar{\theta}}_{..})^2 \tag{8.2.6}$$

$$= k \sum_{i=1}^{p} \sum_{j=1}^{q} \left[|\underline{\bar{\theta}}_{.j}|^2 + |\underline{\bar{\theta}}_{..}|^2 - 2|\underline{\bar{\theta}}_{.j}||\underline{\bar{\theta}}_{..}| \cos(\bar{\theta}_{.j} - \bar{\theta}_{..}) \right]$$

$$= k \sum_{i=1}^{p} \sum_{j=1}^{q} \left[\frac{R^2_{.j}}{N^2_{.j}} + \frac{R^2_{..}}{N^2_{..}} - 2\left[\frac{R_{.j}}{N_{.j}}\right]\left[\frac{R_{..}}{N_{..}}\right] \cos(\bar{\theta}_{.j} - \bar{\theta}_{..}) \right]$$

$$= k \left[\sum_{j=1}^{q} \left[\frac{R^2_{.j}}{N_{.j}}\right] + \frac{R^2_{..}}{N_{..}} - 2\,\frac{R^2_{..}}{N_{..}} \right] \tag{8.2.7}$$

$$\mathrm{BMV} = k \left[\sum_{j=1}^{q} \left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2_{..}}{N_{..}} \right] \tag{8.2.8}$$

Lemma. To show $\sum_{j=1}^{q} R_{.j}\cos(\bar{\theta}_{.j} - \bar{\theta}_{..}) = Nr_{..} = R_{..}$

for equation (8.2.7)

Proof

$$\sum_{j=1}^{q} R_{.j}\cos(\bar{\theta}_{.j} - \bar{\theta}_{..})$$

$$= \sum_{j=1}^{q} N_{.j}r_{.j}(\cos\bar{\theta}_{..}\cos\bar{\theta}_{.j} + \sin\bar{\theta}_{..}\sin\bar{\theta}_{.j})$$

$$= \sum_{j=1}^{q}\left[N_{.j}r_{.j}\left[\frac{\bar{x}_{..}\ \bar{x}_{.j}}{r_{..}\ r_{.j}}\right] + N_{.j}r_{.j}\left[\frac{\bar{y}_{..}\ \bar{y}_{.j}}{r_{..}\ r_{.j}}\right]\right]$$

$$= \left[\frac{\bar{x}_{..}}{r_{..}}\right]N\bar{x}_{..} + \left[\frac{\bar{y}_{..}}{r_{..}}\right]N\bar{y}_{..}$$

$$= \left[\frac{N}{r_{..}}\right]\left[\bar{x}^2_{..} + \bar{y}^2_{..}\right] = Nr_{..} = R_{..}$$

8.3 Cross Product Terms

As discussed in Section 7.4, in standard 'linear' analysis the cross product terms sum to zero, however, with the extended models for circular statistics a non-zero value is obtained. By taking account of the mean resultant lengths this property is re-established. Expression (8.3.1) shows the components within the design

$$k\sum_{i=1}^{p}\sum_{j=1}^{q}(\theta_{ij}-\bar{\theta}_{..})^2 = k\sum_{i=1}^{p}\sum_{j=1}^{q}(\theta_{ij}-\bar{\theta}_{.j}) +$$

(Total) (Residual)

$$k\sum_{i=1}^{p}\sum_{j=1}^{q}(\bar{\theta}_{.j}-\bar{\theta}_{..})^2 - 2k\sum_{i=1}^{p}\sum_{j=1}^{q}(\theta_{ij}-\bar{\theta}_{.j})(\bar{\theta}_{.j}-\bar{\theta}_{..}) \qquad (8.3.1)$$

(Between) (Cross-Product)

From (8.3.1) the cross product term may be broken down as follows;

$$2k\sum_{i=1}^{p}\sum_{j=1}^{q}(\theta_{ij}-\bar{\theta}_{.j})(\bar{\theta}_{.j}-\bar{\theta}_{..}) \qquad (8.3.2)$$

$$= 2k\sum_{i=1}^{p}\sum_{j=1}^{q}\Bigg[|(\underline{\theta}_{ij}||\underline{\bar{\theta}}_{.j}|\cos(\theta_{ij}-\bar{\theta}_{.j})$$

$$- |\underline{\theta}_{ij}||\underline{\bar{\theta}}_{..}|\cos(\theta_{ij}-\bar{\theta}_{..}) - |\underline{\bar{\theta}}_{.j}||\underline{\bar{\theta}}_{.j}| + |\underline{\bar{\theta}}_{.j}||\underline{\bar{\theta}}_{..}|\cos(\bar{\theta}_{.j}-\bar{\theta}_{..})\Bigg]$$

$$= 2k\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\left[\frac{R_{.j}}{N_{.j}}\right]\cos(\theta_{ij}-\bar{\theta}_{.j}) - \left[\frac{R_{..}}{N_{..}}\right]\cos(\theta_{ij}-\bar{\theta}_{..})\right.$$

$$\left. - \left[\frac{R^2_{.j}}{N^2_{.j}}\right] + \left[\frac{R_{.j}}{N_{.j}}\right]\left[\frac{R_{..}}{N_{..}}\right]\cos(\bar{\theta}_{.j}-\bar{\theta}_{..})\right]$$

$$= 2k\left[\sum_{j=1}^{q}\left[\frac{R_{.j}}{N_{.j}}\right] - \frac{R^2_{..}}{N_{..}} - \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] + \frac{R^2_{..}}{N_{..}}\right]$$

$$= 0$$

i.e. The cross product terms sum to zero.

Hence the one-way classification design model can be decomposed as:

$$k\left[N - \frac{R^2_{..}}{N_{..}}\right] = k\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2_{..}}{N_{..}}\right] + k\left[N - \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right]\right] \qquad (8.3.3)$$

(TMV) (BMV) (RMV)

8.4 Analysis of Cross-Classification and Interaction

8.4.1 The Interpretation of Interaction on the Circle

The extent of the problem within the original approach was not fully appreciated until extensions were made to larger designs. Before constructing the statistical test components for larger designs using the approach of Section 8.2 the understanding and interpretation of interaction on the circle will be discussed.

To illustrate the meaning of interaction on the circle a simple example giving population angular means for a particular design may be show. (Table 8.4.1).

Table 8.4.1

		Treatment Level		
		A	B	C
Factor	1	356°	6°	357°
	2	0°	14°	1°

Unlike standard 'linear' statistics a simple subtraction of level mean directions may not be used to indicate the presence of interaction since here periodic values are given and not quantities. The true angle between each of the directions must be calculated in order to observe any differences. (Table 8.4.2).

Table 8.4.2

		Treatment Level				
		A	Diff	B	Diff	C
Factor	1	356°	(10°)	6°	(9°)	357°
	Diff	(4°)		(8°)		(4°)
	2	0°	(14°)	14°	(13°)	1°

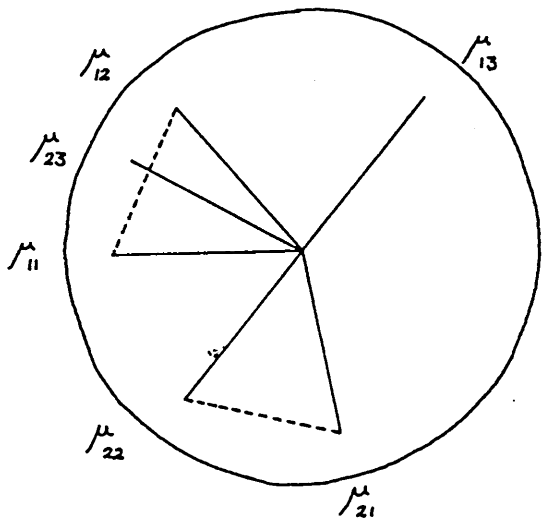
We note that the difference in direction between the two Factor types is larger for the middle level of treatment than for the low and high levels. Similar differences are seen between any two treatment levels and the two Factor types. With unequal differences in sample angular means we may state that interaction exists between factor and treatment levels.

As in linear statistics graphing the cell mean directions can be an aid in interpreting the interaction. For example, consider an experimental design that involves three levels of treatment and two levels of a factor. Lines are used within a unit circle to represent the cell mean directions for the experiment. Table 8.4.3 gives the sample means for the design, with equal concentration parameter within each cell.

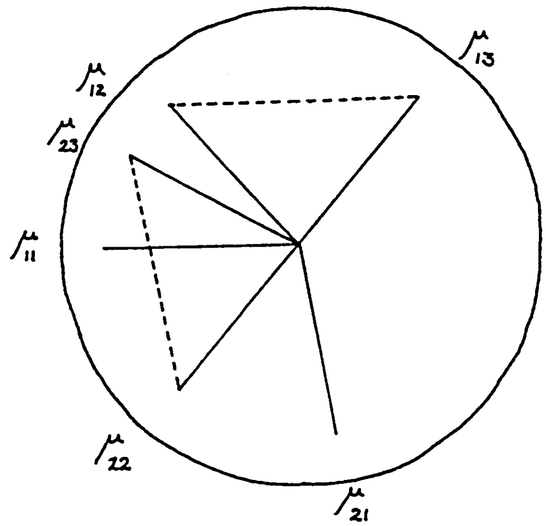
Table 8.4.3

		Treatment Level		
		A	B	C
Factor	1	270°	320°	40°
	2	170°	220°	300°

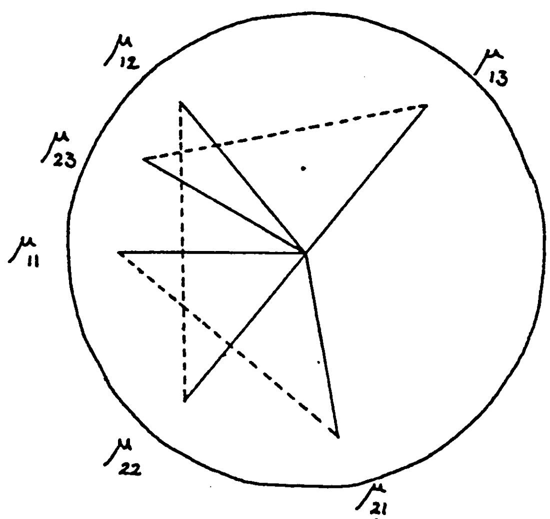
Figure 8.4.1(a) indicates no difference in the length of the line segments between level A and level B of the treatment. Figure 8.4.1(b) shows no difference in the length of line segments between level B and C of the treatment. Similarly Figure 8.4.1(c) shows no difference in the length of the line segments between level 1 and level 2 of the factor. Here Figure 8.4.1 illustrates an experimental design where no interaction is present.



(a) Difference between levels A and B of treatment



(b) Difference between levels B and C of treatment



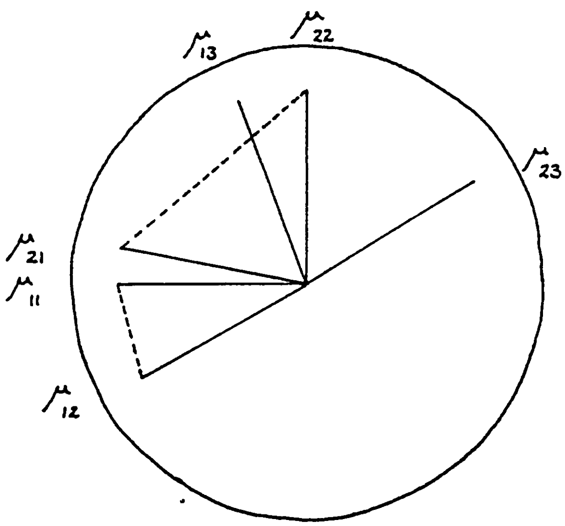
(c) Difference between levels 1 and 2 of factor

Figure 8.4.1 Mean Responses without Interaction

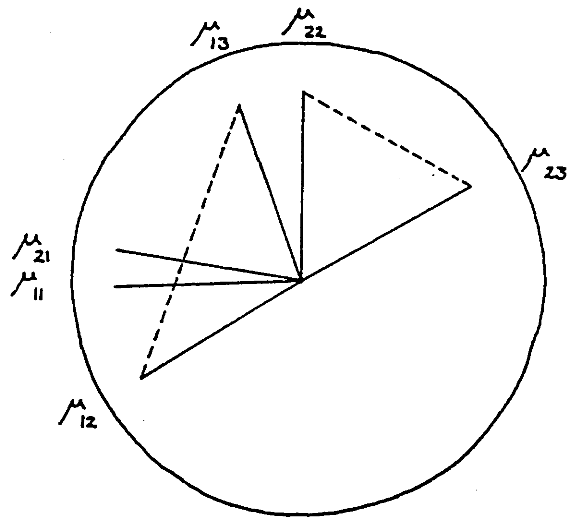
Alternatively Table 8.4.4 and Figure 8.4.2 illustrates the same design layout but with interaction present.

Table 8.4.4

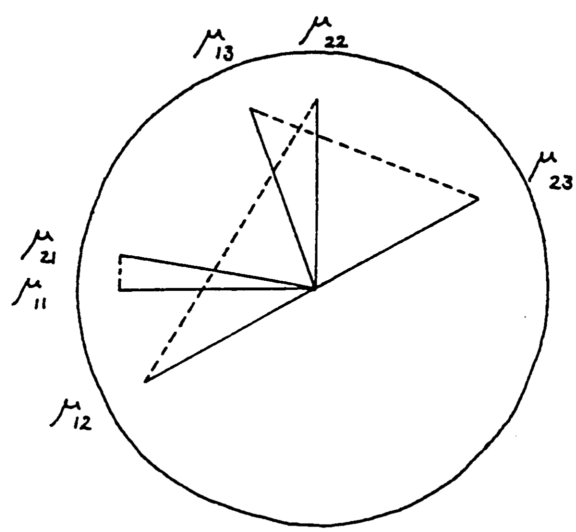
		Treatment Levels		
		A	B	C
Factor	1	270°	240°	340°
	2	280°	360°	60°



(a) Difference between levels A and B of treatment



(b) Difference between levels B and C of treatment



(c) Difference between levels 1 and 2 of factor

Figure 8.4.2 Mean Responses with Interaction

As in 'linear' statistics when the data indicates that large interactions exist, it is important to consider whether large interactions actually are present in the sample means or whether there may be some other explanation for the occurrence of the interactions in the data.

Unexpected interactions may be caused by a problem in the data, there may be an outlier or an erroneous response. Possibly another effect may be taking place which has not been accounted for. In experiments involving animals, for example, where room or time of day may give an apparent interaction when none exists. In such a case, the errors can no longer be said to be random or independent. Thus an unexpected interaction may be a clue to a failure in meeting the assumptions of the model being used.

8.4.2 The Calculation of Interaction on the Circle

The between cells sum of squares in 'linear' statistics can be split up to produce a between rows, between columns and interaction sum of squares. For a two way analysis the individual cell interaction values are made up of three components, the distance from the overall mean to the cell mean $(\bar{x}_{...} - \bar{x}_{ij.})$, from the row mean to to the cell mean $(\bar{x}_{i..} - \bar{x}_{ij.})$, and from the column mean to the cell mean $(\bar{x}_{.j.} - \bar{x}_{ij.})$.

$$(\bar{x}_{...} - \bar{x}_{ij.}) - (\bar{x}_{i..} - \bar{x}_{ij.}) - (\bar{x}_{.j.} - \bar{x}_{ij.})$$

Giving

$$\bar{x}_{ij.} - \bar{x}_{i..} - \bar{x}_{.j.} + \bar{x}_{...} \tag{8.4.1}$$

For the overall interaction this distance is squared and summed over all observations. For interaction on the circle Figure 8.4.3 illustrates the sample mean direction for the three vector lengths involved.

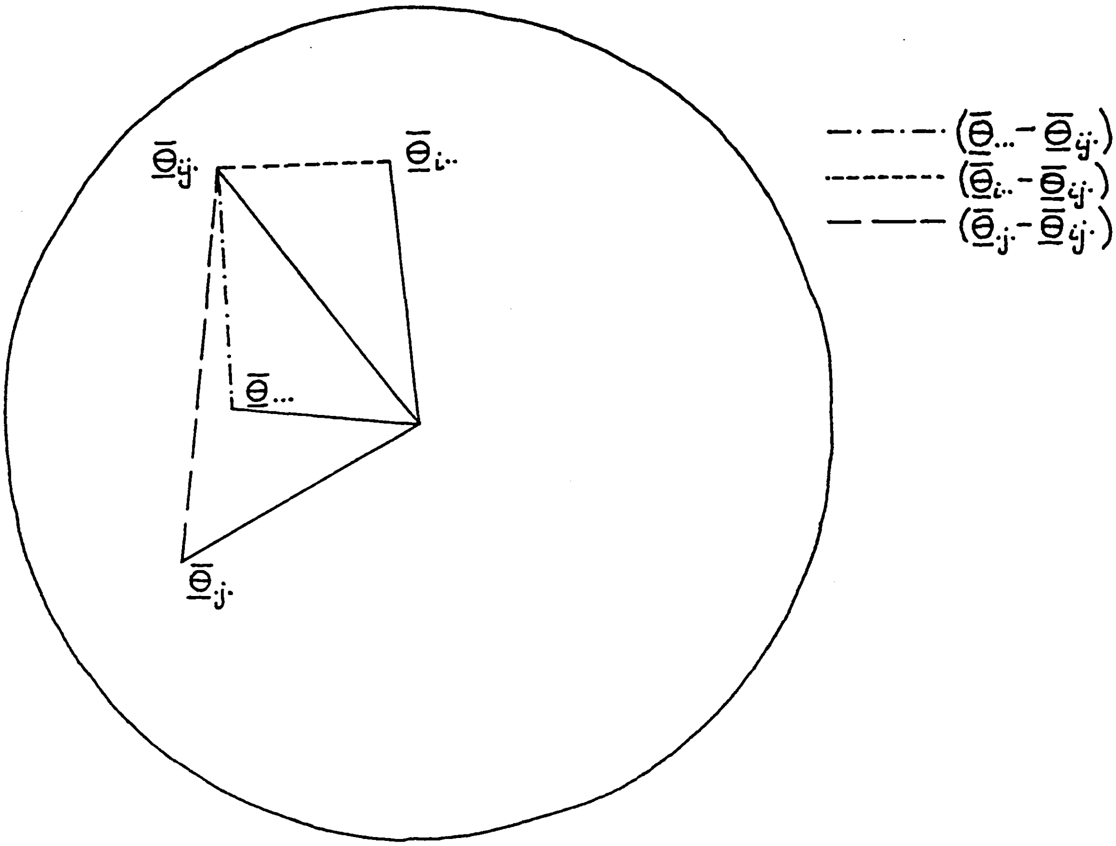


Figure 8.4.3 The Components of Interaction

Interaction on the circle may then be given by;

$$k \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{l=1}^{m} (\underline{\bar{\theta}}_{ij.} - \underline{\bar{\theta}}_{i..} - \underline{\bar{\theta}}_{.j.} - \underline{\bar{\theta}}_{...})^2 \tag{8.4.2}$$

$$= k \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{l=1}^{m} \left[\frac{R^2_{ij.}}{N^2_{ij.}} + \frac{R^2_{i..}}{N^2_{i..}} + \frac{R^2_{.j.}}{N^2_{.j.}} + \frac{R^2_{...}}{N^2_{...}} \right.$$

$$- 2\left[\frac{R_{ij.}}{N_{ij.}}\right]\left[\frac{R_{i..}}{N_{i..}}\right]\cos(\bar{\theta}_{ij.} - \bar{\theta}_{i..}) - 2\left[\frac{R_{ij.}}{N_{ij.}}\right]\left[\frac{R_{.j.}}{N_{.j.}}\right]\cos(\bar{\theta}_{ij.} - \bar{\theta}_{.j.})$$

$$- 2\left[\frac{R_{i..}}{N_{i..}}\right]\left[\frac{R_{...}}{N_{...}}\right]\cos(\bar{\theta}_{i..} - \bar{\theta}_{...}) - 2\left[\frac{R_{.j.}}{N_{.j.}}\right]\left[\frac{R_{...}}{N_{...}}\right]\cos(\bar{\theta}_{.j.} - \bar{\theta}_{...})$$

$$+ 2\left[\frac{R_{ij.}}{N_{ij.}}\right]\left[\frac{R_{...}}{N_{...}}\right]\cos(\bar{\theta}_{ij.} - \bar{\theta}_{...}) + 2\left[\frac{R_{i..}}{N_{i..}}\right]\left[\frac{R_{.j.}}{N_{.j.}}\right]\cos(\bar{\theta}_{i..} - \bar{\theta}_{.j.})\Bigg]$$

(8.4.3)

It can be shown that

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{l=1}^{m}\left[\frac{R_{ij.}}{N_{ij.}}\right]\left[\frac{R_{i..}}{N_{i..}}\right]\cos(\bar{\theta}_{ij.} - \bar{\theta}_{i..}) = \sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i..}}\right]$$

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{l=1}^{m}\left[\frac{R_{ij.}}{N_{ij.}}\right]\left[\frac{R_{.j.}}{N_{.j.}}\right]\cos(\bar{\theta}_{ij.} - \bar{\theta}_{.j.}) = \sum_{j=1}^{q}\left[\frac{R^2_{.j.}}{N_{.j.}}\right]$$

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{l=1}^{m}\left[\frac{R_{.j.}}{N_{.j.}}\right]\left[\frac{R_{...}}{N_{...}}\right]\cos(\bar{\theta}_{.j.} - \bar{\theta}_{...}) = \frac{R^2_{...}}{N_{...}}$$

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{l=1}^{m}\left[\frac{R_{i..}}{N_{i..}}\right]\left[\frac{R_{...}}{N_{...}}\right]\cos(\bar{\theta}_{i..} - \bar{\theta}_{...}) = \frac{R^2_{...}}{N_{...}}$$

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{l=1}^{m}\left[\frac{R_{ij.}}{N_{ij.}}\right]\left[\frac{R_{...}}{N_{...}}\right]\cos(\bar{\theta}_{ij.} - \bar{\theta}_{...}) = \frac{R^2_{...}}{N_{...}}$$

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{l=1}^{m}\left[\frac{R_{i..}}{N_{i..}}\right]\left[\frac{R_{.j.}}{N_{.j.}}\right]\cos(\bar{\theta}_{i..} - \bar{\theta}_{.j.}) = \frac{R^2_{...}}{N_{...}}$$

Replacing these into (8.4.3) gives

$$k\left[\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R^2_{ij.}}{N_{ij.}}\right] - \sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i..}}\right] - \sum_{j=1}^{q}\left[\frac{R^2_{.j.}}{N_{.j.}}\right] + \frac{R^2_{...}}{N_{...}}\right] \qquad (8.4.4)$$

A comparable breakdown of the between measure of variation for the circle can now be seen as

$$k\left[\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R^2_{ij.}}{N_{ij.}}\right] - \frac{R^2_{...}}{N_{...}}\right] = k\left[\sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i..}}\right] - \frac{R^2_{...}}{N_{...}}\right] + k\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j.}}{N_{.j.}}\right] - \frac{R^2_{...}}{N_{...}}\right]$$

(Between Factor 1 or Rows) (Between Factor 2 or Columns)

$$+ k\left[\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R^2_{ij.}}{N_{ij.}}\right] - \sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i..}}\right] - \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j.}}\right] + \frac{R^2_{...}}{N_{...}}\right] \tag{8.4.5}$$

(Interaction)

In the same manner as Section 7.4 the cross product terms may be found for the two-way classification with interaction design. Following lengthy vector algebra we may show that the cross product terms listed below all equal zero.

$$2k\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{l=1}^{m}(\underline{\bar{\theta}}_{i..} - \underline{\bar{\theta}}_{...})(\underline{\bar{\theta}}_{.j.} - \underline{\bar{\theta}}_{...}) = 0$$

$$2k\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{l=1}^{m}(\underline{\bar{\theta}}_{i..} - \underline{\bar{\theta}}_{...})(\underline{\bar{\theta}}_{ij.} - \underline{\bar{\theta}}_{i..} - \underline{\bar{\theta}}_{.j.} + \underline{\bar{\theta}}_{...}) = 0$$

$$2k\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{l=1}^{m}(\underline{\bar{\theta}}_{i..} - \underline{\bar{\theta}}_{...})(\underline{\bar{\theta}}_{.jl} - \underline{\bar{\theta}}_{ij.}) = 0$$

$$2k\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{l=1}^{m}(\underline{\bar{\theta}}_{.j.} - \underline{\bar{\theta}}_{...})(\underline{\bar{\theta}}_{ij.} - \underline{\bar{\theta}}_{i..} - \underline{\bar{\theta}}_{.j.} + \underline{\bar{\theta}}_{...}) = 0$$

$$2k\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{l=1}^{m}(\underline{\bar{\theta}}_{.j.} - \underline{\bar{\theta}}_{...})(\underline{\bar{\theta}}_{.jl} - \underline{\bar{\theta}}_{ij.}) = 0$$

$$2k\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{l=1}^{m}(\underline{\bar{\theta}}_{ijl} - \underline{\bar{\theta}}_{ij.})(\underline{\bar{\theta}}_{ij.} - \underline{\bar{\theta}}_{i..} - \underline{\bar{\theta}}_{.j.} + \underline{\bar{\theta}}_{...}) = 0$$

Finally the two-way analysis of variance model with interaction may be given as

$$k\left[N - \frac{R^2_{...}}{N_{...}}\right] = k\left[\sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i..}}\right] - \frac{R^2_{...}}{N_{...}}\right] + k\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j.}}{N_{.j.}}\right] - \frac{R^2_{...}}{N_{...}}\right]$$

$$+ k\left[\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R^2_{ij.}}{N_{ij.}}\right] - \sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i..}}\right] - \sum_{j=1}^{q}\left[\frac{R^2_{.j.}}{N_{.j.}}\right] + \frac{R^2_{...}}{N_{...}}\right]$$

$$+ k\left[N - \sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R^2_{ij.}}{N_{ij.}}\right]\right] \tag{8.4.6}$$

8.5 Other Design Models

8.5.1 The Randomised Complete Block Design

Section 6.3 discussed the requirement and structure of the randomised complete block design where the blocks are formed so that each is as homogeneous as possible. Within this design no interaction exists. The vector difference $(\theta_{ij} - \overline{\theta}_{..})$ i.e. the total measure of variation, may be expressed as the sum of three terms

$$(\underline{\theta}_{ij} - \underline{\overline{\theta}}_{..}) = (\underline{\overline{\theta}}_{i.} - \underline{\overline{\theta}}_{..}) + (\underline{\overline{\theta}}_{.j} - \underline{\overline{\theta}}_{..}) + (\underline{\theta}_{ij} - \underline{\overline{\theta}}_{i.} - \underline{\overline{\theta}}_{.j} + \underline{\overline{\theta}}_{..}) \tag{8.5.1}$$

Both the first two terms on the right hand side of (8.5.1) have been seen in Section 7.3.3 and give

$$k\sum_{i=1}^{p}\sum_{j=1}^{q}(\underline{\overline{\theta}}_{i.} - \underline{\overline{\theta}}_{..})^2 = k\left[\sum_{i=1}^{p}\left[\frac{R^2_{i.}}{N_{i.}}\right] - \frac{R^2_{..}}{N_{..}}\right] \tag{8.5.2}$$

$$k\sum_{i=1}^{p}\sum_{j=1}^{q}(\underline{\overline{\theta}}_{.j} - \underline{\overline{\theta}}_{..})^2 = k\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2_{..}}{N_{..}}\right] \tag{8.5.3}$$

Note that (8.5.2) provides an estimate of the measure of variation between p treatments and that (8.5.3) provides an estimate of the measure of variation between q blocks. The third term on the right hand side of (8.5.1) is an estimate of the residual measure within the design. Using the same procedure as Sections 8.2 and 8.4 it can be shown that

$$k\sum_{i=1}^{p}\sum_{j=1}^{q}(\underline{\theta}_{ij} - \underline{\bar{\theta}}_{i.} - \underline{\bar{\theta}}_{.j} + \underline{\bar{\theta}}_{..})^2 = k\left[N - \sum_{i=1}^{p}\left[\frac{R_{i.}^2}{N_{i.}}\right] - \sum_{j=1}^{q}\left[\frac{R_{.j}^2}{N_{.j}}\right] + \frac{R_{..}^2}{N_{..}}\right] \tag{8.5.4}$$

In the same manner as for the two way classification with interaction the cross product terms all equal zero. Hence the randomised complete block design may be given as

$$k\left[N - \frac{R_{..}^2}{N_{..}}\right] = k\left[\sum_{i=1}^{p}\left[\frac{R_{i.}^2}{N_{i.}}\right] - \frac{R_{..}^2}{N_{..}}\right] + k\left[\sum_{j=1}^{q}\left[\frac{R_{.j}^2}{N_{.j}}\right] - \frac{R_{..}^2}{N_{..}}\right]$$

$$+ k\left[N - \sum_{i=1}^{p}\left[\frac{R_{i.}^2}{N_{i.}}\right] - \sum_{j=1}^{q}\left[\frac{R_{.j}^2}{N_{.j}}\right] + \frac{R_{..}^2}{N_{..}}\right] \tag{8.5.5}$$

8.5.2 Latin Squares Design

In the randomised complete block design, the effect of a single factor was removed. It is occasionally possible to eliminate two sources of non-homogeneity simultaneously in the same experiment by using the Latin square design. Such designs were originally applied in agricultural experimentation when the two directional sources of non-homogeneity were simply the two directions on the field, and the "square" was literally a square plot of land. Its usage has been extended to many other applications where there are two sources of non-homogeneity that may affect experimental results, for example, machines, positions, operators, runs, days. A third variable, the experimental treatment, is then associated with the two source variables in a prescribed fashion. The use of Latin squares is restricted by two conditions:

(i) the number of rows, columns and treatments must all be the same

(ii) there must be no interaction between row and column factors

The analysis of Latin squares is based on essentially the same assumptions as the analysis of randomised blocks. The essential difference is that in the case of randomised blocks we allow for one source of non-homogeneity (represented by blocks) while in the case of Latin squares we are simultaneously allowing for two kinds of non-homogeneity (represented by rows and columns). As with the randomised complete block design the relatively simple but lengthy proof of construction via vector algebra has not been reiterated, however, all cross product terms can be shown to equal zero, producing the model expression:-

$$k\left[N - \frac{R^2_{...}}{N_{...}}\right] = k\left[\sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i..}}\right] - \frac{R^2_{...}}{N_{...}}\right] + k\left[\sum_{j=1}^{p}\left[\frac{R^2_{.j.}}{N_{.j.}}\right] - \frac{R^2_{...}}{N_{...}}\right]$$

$$+ k\left[\sum_{l=1}^{p}\left[\frac{R^2_{..l}}{N_{..l}}\right] - \frac{R^2_{...}}{N_{...}}\right]$$

$$+ k\left[N - \sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i..}}\right] - \sum_{j=1}^{p}\left[\frac{R^2_{.j.}}{N_{.j.}}\right] - \sum_{l=1}^{p}\left[\frac{R^2_{..l}}{N_{..l}}\right] + 2\frac{R^2_{...}}{N_{...}}\right] \qquad (8.5.6)$$

8.5.3 Nested or Hierarchical Design

Section 6.2 discussed the structure of the nested design; as with the randomised complete block and Latin square designs the nested design may also be constructed in the same manner to produce the model expression;

$$k\left[N - \frac{R^2_{\cdots}}{N_{\cdots}}\right] = k\left[\sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i..}}\right] - \frac{R^2_{\cdots}}{N_{\cdots}}\right] + k\left[\sum_{j=1}^{q_1}\left[\frac{R^2_{1j.}}{N_{1j.}}\right] - \frac{R^2_{1..}}{N_{1..}}\right]$$

$$+ k\left[\sum_{j=1}^{q_2}\left[\frac{R^2_{2j.}}{N_{2j.}}\right] - \frac{R^2_{2..}}{N_{2..}}\right] + \cdots\cdots$$

$$+ k\left[\sum_{j=1}^{q_p}\left[\frac{R^2_{pj.}}{N_{pj.}}\right] - \frac{R^2_{p..}}{N_{p..}}\right] + k\left[N - \sum_{i=1}^{p}\sum_{j=1}^{q_i}\left[\frac{R^2_{ij.}}{N_{ij.}}\right]\right] \qquad (8.5.7)$$

8.5.4 Three–Way Classification

Clearly larger and larger designs can be constructed in the above manner, here a three–way classification design has been built to illustrate the generalised nature of the approach;

$$
\begin{aligned}
& k\left[N - \frac{R^2_{\cdot\cdot\cdot}}{N_{\cdot\cdot\cdot}}\right] - k\left[\sum_{i=1}^{p}\left[\frac{R^2_{i\cdot\cdot}}{N_{i\cdot\cdot}}\right] - \frac{R^2_{\cdot\cdot\cdot}}{N_{\cdot\cdot\cdot}}\right] + k\left[\sum_{j=1}^{q}\left[\frac{R^2_{\cdot j\cdot}}{N_{\cdot j\cdot}}\right] - \frac{R^2_{\cdot\cdot\cdot}}{N_{\cdot\cdot\cdot}}\right] \\
& \quad + k\left[\sum_{l=1}^{m}\left[\frac{R^2_{\cdot\cdot l}}{N_{\cdot\cdot l}}\right] - \frac{R^2_{\cdot\cdot\cdot}}{N_{\cdot\cdot\cdot}}\right] \\
& \quad + k\left[\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R^2_{ij\cdot}}{N_{ij\cdot}}\right] - \sum_{i=1}^{p}\left[\frac{R^2_{i\cdot\cdot}}{N_{i\cdot\cdot}}\right] - \sum_{j=1}^{q}\left[\frac{R^2_{\cdot j\cdot}}{N_{\cdot j\cdot}}\right] + \frac{R^2_{\cdot\cdot\cdot}}{N_{\cdot\cdot\cdot}}\right] \\
& \quad + k\left[\sum_{i=1}^{p}\sum_{l=1}^{m}\left[\frac{R^2_{i\cdot l}}{N_{i\cdot l}}\right] - \sum_{i=1}^{p}\left[\frac{R^2_{i\cdot\cdot}}{N_{i\cdot\cdot}}\right] - \sum_{l=1}^{m}\left[\frac{R^2_{\cdot\cdot l}}{N_{\cdot\cdot l}}\right] + \frac{R^2_{\cdot\cdot\cdot}}{N_{\cdot\cdot\cdot}}\right] \\
& \quad + k\left[\sum_{j=1}^{q}\sum_{l=1}^{m}\left[\frac{R^2_{\cdot jl}}{N_{\cdot jl}}\right] - \sum_{j=1}^{q}\left[\frac{R^2_{\cdot j\cdot}}{N_{\cdot j\cdot}}\right] - \sum_{l=1}^{m}\left[\frac{R^2_{\cdot\cdot l}}{N_{\cdot\cdot l}}\right] + \frac{R^2_{\cdot\cdot\cdot}}{N_{\cdot\cdot\cdot}}\right] \\
& \quad + k\left[\sum_{i=1}^{p}\sum_{l=1}^{m}\sum_{j=1}^{q}\left[\frac{R^2_{ijl}}{N_{ijl}}\right] - \sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R^2_{ij\cdot}}{N_{ij\cdot}}\right] - \sum_{j=1}^{q}\sum_{l=1}^{m}\left[\frac{R^2_{\cdot jl}}{N_{\cdot jl}}\right]\right. \\
& \quad - \sum_{i=1}^{p}\sum_{l=1}^{m}\left[\frac{R^2_{i\cdot l}}{N_{i\cdot l}}\right] + \sum_{i=1}^{p}\left[\frac{R^2_{i\cdot\cdot}}{N_{i\cdot\cdot}}\right] + \sum_{j=1}^{q}\left[\frac{R^2_{\cdot j\cdot}}{N_{\cdot j\cdot}}\right] \\
& \quad \left. + \sum_{l=1}^{m}\left[\frac{R^2_{\cdot\cdot l}}{N_{\cdot\cdot l}}\right] - \frac{R^2_{\cdot\cdot\cdot}}{N_{\cdot\cdot\cdot}}\right] \\
& \quad + k\left[N - \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{l=1}^{m}\left[\frac{R^2_{ijl}}{N_{ijl}}\right]\right] \qquad (8.5.8)
\end{aligned}
$$

Chapter 11 shows further use of this generalised method with the implimentation of a split plot design to real data arising in an angular form.

8.6 Summary

It was seen in Chapter 7 how the combining of angular mean directions may give a false overall mean direction. Taking account of the mean resultant lengths together with their corresponding mean directions has been shown to eliminate this problem and has helped to indicate a new approach to the circular analysis of variance. It has been emphasised that the approach is, in the first instance, an analysis of differing populations, via the maximum likelihood ratio test, rather than differing mean directions. The requirement for the testing of the equality of concentration parameters is essential in order that a true test of mean directions may be undertaken.

Using the knowledge of angular mean combinations, the use of vector algebra and directional data properties, new components for the total, residual and between measures of variation have been built. The significance of this method has shown how the cross product terms, found to be non-zero for the original approach in Chapter 7 and requiring a correction factor, are now zero.

This method has then been extended to discuss, explain and illustrate the construction of interaction on the circle and hence build the new procedure for the two-way classification design, showing zero cross product terms.

Finally this generalised method has been further extended to construct other larger designs such as the Latin square and three-way classification design.

CHAPTER 9

THE DISTRIBUTION OF THE NEW COMPONENTS AND TEST STATISTICS

9.1 Introduction

Having developed the generalised procedures in Chapter 8 it is necessary to examine the theory of the associated statistical tests for all experimental situations which may occur. Some attempt was made to evaluate theoretically the exact distributions of the test statistics. However, as was found by Upton (1972) and Stephens (1969, 72), even for simple single sample tests the numerical integration involved was extremely tedious. R does not have a simple density function and a direct evaluation of the significance points is not straightforward. Stephens (1969) gave the upper and lower 1% and 5% points for several values of k and N for R/N and X/N. Stephens (1972) has also evaluated the exact two-sample test given by equation (4.2.2) for differing values of k and N. In 1969 Stephens discussed the problem of obtaining the exact theoretical distribution of

$$\left[\sum_{j=1}^{q} R_{.j} - R_{..}\right]$$

for the sphere and circle developed by Watson (1956) and Watson and Williams (1956). Stephens also stated that the analysis is not so straightforward as for the Normal distribution; with the distribution of the test statistic being intractable. This, unfortunately, is also true for the distribution of the test statistic

$$\sum_{j=1}^{q}\left[\frac{R_{.j}^2}{N_{.j}}\right] - \frac{R_{..}^2}{N}$$

from (8.3.3). However, the asymptotic results, as for the test statistic by Watson and Williams, may be investigated.

Sections 9.2 and 9.3 will examine the component statistics for small and large k respectively. In addition the first two moments for each of the components are found and used to improve the associated approximation. The accuracy of the chi-squared distributions and their corresponding F statistics are examined via the same simulation techniques used in Chapter 6.

As with standard 'linear' analysis of variance, it is important to reiterate that because of the way in which the new procedures are constructed, the equality of the concentration parameters are examined prior to any analysis of variance.

9.2 Small Concentration Parameter, k

As given in Chapter 8 the new procedure is based on the likelihood ratio test for the null hypothesis $\beta_1 = ... = \beta_q$, $k_1 = ... = k_q$ against its general alternative hypothesis. Let λ be the likelihood ratio criterion for this problem, as in (3.1.5), giving the test criterion

$$-2\ \log\lambda = 2\left[\sum_{j=1}^{q} \hat{k}_j R_{.j} - \hat{k}_0 R + \sum_{j=1}^{q} N_{.j} \log\left\{\frac{I_0(\hat{k}_0)}{I_0(\hat{k}_j)}\right\}\right] \tag{9.2.1}$$

where $\hat{k}_0$ and $\hat{k}_j$ are given by

$$A(\hat{k}_0) = \left[\frac{R}{N}\right] \qquad A(\hat{k}_j) = \left[\frac{R_{.j}}{N_{.j}}\right] \quad j = 1,2,\ldots,q \tag{9.2.2}$$

The power series for the ratio of Bessel functions $I_0(k)$ and $I_1(k)$ for small k are given in (3.2.8). The first term approximation, $A(k) = k/2$, has been shown to be tolerable for $k \lessapprox 1$. Using this in (9.2.2) gives

$$\hat{k}_0 = \left[\frac{2R}{N}\right] \qquad \hat{k}_j = \left[\frac{2R_{.j}}{N_{.j}}\right] \quad j = 1,2,\ldots,q \tag{9.2.3}$$

also given in (3.2.11) as approximation $\hat{k}_7$. Hence for small k (9.2.1) reduces to

$$-2 \log\lambda = 2\left[\sum_{j=1}^{q}\left[\frac{R^2_{\cdot j}}{N_{\cdot j}}\right] - \frac{R^2}{N}\right] \approx \chi^2_{2(q-1)} \tag{9.2.4}$$

This same procedure is given by Mardia (1972) for his S statistic. Identity (9.2.4) is seen as the between measure of variation (8.2.8), for small k.

From Rayleigh (1919) and Lord (1954) it has been shown that $2R^2/N$ has a chi-squared distribution with 2 degrees of freedom. This result was given by Rayleigh after finding the form of the p.d.f. of R, equation (2.3.14), for large N. This result has been used to test whether the population from which a sample is drawn differs significantly from uniformity. We cannot, however, show $2(N-R^2/N)$ to be chi-squared in a similar manner. Following preliminary investigations and simulation routines both $2(N-R^2/N)$ and

$$2\left[N - \sum_{j=1}^{q}\left[\frac{R^2_{\cdot j}}{N_{\cdot j}}\right]\right]$$

are found to be what may be termed as negative or reflected chi-squared distributions (or random variables), negatively rather than positively skewed. With the inclusion of N the chi-squared is effectively transformed to a reflected chi-squared with its variance decreasing as k increases.

For small k we may only investigate the between measure as produced from maximum likelihood. This chi-squared may be improved following the equating of expectations, using

$$E(R^2_{\cdot j}) = N_{\cdot j} + N_{\cdot j}(N_{\cdot j} - 1)\rho^2 \qquad \text{from (2.5.2)}$$

where $\rho = A(k) = \dfrac{I_1(k)}{I_0(k)}$ and

$$E(\chi^2_{2(q-1)}) = 2(q - 1)$$

Producing the improvement factor

$$\gamma = \frac{1}{1 - \rho^2} \qquad (9.2.5)$$

Therefore we may approximate the distribution

$$\frac{2}{1 - \rho^2}\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2}{N}\right] \qquad (9.2.6)$$

by a χ^2 variable with 2(q–1) degrees of freedom. When k is unknown, the maximum likelihood estimator $\hat{k}$, given by (3.2.2), will be used.

9.3 Large Concentration Parameter, k

Watson and Williams showed that $2k(N-R)$ is approximately distributed as a chi–squared with (N–1) degrees of freedom. For the procedure discussed in Chapter 8 we are required to show that $k(N-R^2/N)$ is distributed as chi–squared with (N–1) degrees of freedom, for large k. As a simple initial proof we may show that

$$k\left[N - \frac{R^2}{N}\right] = k(N - R)\left[1 + \frac{R}{N}\right]$$

For large k, $R/N \rightarrow 1$, therefore

$$k\left[N - \frac{R^2}{N}\right] \rightarrow 2k(N - R)$$

Alternatively we may adapt the approach of Mardia (1972, p 114) where the distributions of $2k(N-C)$, $2k(R-C)$ and $2k(N-R)$ are found. Here the distributions of $k(N-C^2/N))$, $k((R^2/N)-(C^2/N))$ and $k(N-(R^2/N))$ are required.

9.3.1 Distribution of $k(N - C^2/N)$

Let θ be distributed as $M(0,k)$, then for large k we wish to show that

$$\frac{1}{N}(N - C)k(N + C) = \chi^2_N \tag{9.3.1}$$

Let

$$\epsilon = \sum_{j=1}^{N} k(1 - \cos\theta_{.j})(1 + \cos\theta_{.j})$$

$$= \sum_{j=1}^{N} k\left[1 - \left[1 - \frac{\theta^2_{.j}}{2!} + \frac{\theta^4_{.j}}{4!} - \ldots\ldots\right]\right]\left[1 + \left[1 - \frac{\theta^2_{.j}}{2!} + \frac{\theta^4_{.j}}{4!} - \ldots\ldots\right]\right]$$

$$= \sum_{j=1}^{N} k\left[\theta^2_{.j} - \frac{2\theta^4_{.j}}{4!} - \frac{\theta^4_{.j}}{2!2!} - \frac{\theta^6_{.j}}{4!2!} - \ldots\ldots\right]$$

For large k, $\theta_{.j}$ is small

$$\epsilon \approx \sum_{j=1}^{N} k\theta^2_{.j}$$

From (2.2.2) $\theta_{.j}$ may be approximated by $N(0,k^{-\frac{1}{2}})$ and therefore $k\theta_{.j}{}^2$ will be distributed as the square of the standard normal variate which is approximately a chi-squared distribution with 1 degree of freedom. Hence, by the additive property of chi-squared

$$\sum_{j=1}^{N} k\theta^2_{.j} \approx k\left[N - \frac{C^2}{N}\right] \approx \chi^2_N \tag{9.3.2}$$

9.3.2 Distribution of $k((R^2/N) - (C^2/N))$

It can be seen that

$$\frac{k}{N}(R + C)(R - C) = k\frac{R^2}{N}(1 - \cos\bar{\theta})(1 + \cos\bar{\theta}) \qquad (9.3.3)$$

It has been shown by Mardia (1972, p 98) that the conditional distribution of $\bar{\theta}$ (sample mean direction) given R is $M(\theta, kR)$. On using (9.3.1) the conditional distribution of (9.3.3) for given R is χ_1^2 which does not depend on R.

9.3.3 Distribution of $k(N - (R^2/N))$

Following the identity

$$k\left[N - \frac{C^2}{N}\right] = k\left[\frac{R^2}{N} - \frac{C^2}{N}\right] + k\left[N - \frac{R^2}{N}\right] \qquad (9.3.4)$$

and using (9.3.2) and (9.3.3), by the additive property of chi-squared it can be shown that

$$k\left[N - \frac{R^2}{N}\right] \approx \chi^2_{N-1} \qquad (9.3.5)$$

where $k((R^2/N)-(C^2/N))$ and $k(N-(R^2/N))$ are independently distributed.

In the same manner as Watson and Williams original expression, (9.3.4) behaves like the similar form found in 'linear' analysis of variance.

Using a similar approach, it follows that for large k

$$k\left[N - \left[\frac{R^2}{N}\right]\right] \approx \chi^2_{N-1} \qquad k\left[N_{.j} - \left[\frac{R^2_{.j}}{N_{.j}}\right]\right] \approx \chi^2_{N_{.j}-1} \qquad (9.3.6)$$

Therefore

$$k\left[N - \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right]\right] \approx \chi^2_{N-q} \qquad k\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2}{N}\right] \approx \chi^2_{q-1} \tag{9.3.7}$$

As for small k these approximations may be improved following the use of expectation (2.5.2). For large k, equating expectations gives an improvement factor

$$\gamma = \frac{1}{k(1 - \rho^2)} \tag{9.3.8}$$

Therefore an improvement, when k is unknown, is made by replacing k by (9.2.5)

$$\frac{1}{1 - \rho^2}\left[N - \frac{R^2}{N}\right] \approx \chi^2_{N-1} \tag{9.3.9}$$

The remaining components also require the same improvement factor, giving;

$$\frac{1}{1 - \rho^2}\left[N - \frac{R^2}{N}\right] = \frac{1}{1 - \rho^2}\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2}{N}\right] + \frac{1}{1 - \rho^2}\left[N - \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right]\right] \tag{9.3.10}$$

with associated chi-squared distributions

$$\chi^2_{N-1} = \chi^2_{q-1} + \chi^2_{N-q}$$

9.4 The Variance of the Component Chi-Squared Approximations

9.4.1 Variance of $\left[N - \frac{R^2}{N}\right]$

$$\mathrm{Var}\left[N - \frac{R^2}{N}\right] = E\left[N - \frac{R^2}{N}\right]^2 - \left[E\left[N - \frac{R^2}{N}\right]\right]^2$$

$$= E(N^2) - 2E(R^2) + E\left[\frac{R^4}{N^2}\right] - [(N - 1)^2 - 2(N - 1)^2\rho^2 + (N - 1)^2\rho^4]$$

Using expectations $E(R^2)$ and $E(R^4)$ given by (2.5.2) and (2.5.3)

$$= 2(1 - N)\rho^2 - 1 - (N - 1)^2\rho^4 + \frac{1}{N^2}\left[\frac{N!}{(N - 4)!}\rho^4 \right.$$

$$\left. + \frac{2N!}{(N - 3)!}\rho^2(2 + \rho_2) + \frac{N!}{(N - 2)!}(2 + 4\rho^2 + \rho_2^2) + N \right] \qquad (9.4.1)$$

Let (9.4.1) be represented by T. For large k, equating T to the variance of its associated chi-squared

$$k^2 T = 2(N - 1) \qquad (9.4.2)$$

Let S be the improvement for variance, therefore

$$S^2 = \frac{2(N - 1)}{k^2 T} \qquad (9.4.3)$$

Re-equating expectations gives

$$S\left[k\left[N - \frac{R^2}{N}\right]\right] + (N - 1)[1 - Sk(1 - \rho^2)] \approx \chi^2_{N-1} \qquad (9.4.4)$$

This lengthy expression has been tested on simulations for large k ($\geqslant$ 2) and excellent chi-squared approximations produced. However, with the adjustment for the expectation given in (9.4.4) it is possible for negative values to be obtained. Similarly, as part of an analysis of variance procedure (9.4.4) would be impracticable.

In order to study the variance (9.4.1) further, the asymptotic expansion of ρ has been used. For reasonably accurate approximations the first four terms of the series A(k) are required; for large k

$$\left.\begin{aligned}
\rho &= 1 - \frac{1}{2k} - \frac{1}{8k^2} - \frac{1}{8k^3} \qquad \text{from (3.3.2)} \\
\rho^2 &= 1 - \frac{1}{k} - \frac{1}{8k^3} + \frac{9}{64k^4} + \frac{1}{32k^5} + \frac{1}{64k^6} \\
\rho_2 &= 1 - \frac{2}{k} + \frac{1}{k^2} + \frac{1}{4k^3} + \frac{1}{4k^4}
\end{aligned}\right\} \qquad (9.4.5)$$

When these equations and those of p_2^2 and p_4 are substituted into (9.4.1), the variance of the chi-squared approximation,

$$\mathrm{Var}\left[N - \frac{R^2}{N}\right] = \frac{1}{k^2}(2N - 2) + \frac{1}{k^3}\left[\frac{1}{N} - N\right] + \frac{1}{k^4}\left[\frac{-53}{32N} + \frac{89}{32} - \frac{5N}{32}\right]$$

$$+ \frac{1}{k^5}\left[\frac{-9}{16N} + \frac{7}{8} - \frac{5N}{16}\right] + \frac{1}{k^6}\left[\frac{-1}{16N} - \frac{7}{32} + \frac{9N}{32}\right] + 0\left[\frac{1}{k^7}\right] \qquad (9.4.6)$$

Therefore

$$\mathrm{Var}\left\{k\left[N - \frac{R^2}{N}\right]\right\} = 2(N - 1) + \frac{1}{k}\left[\frac{1}{N} - N\right] + \ldots\ldots \qquad (9.4.7)$$

This interesting result shows that the variance of the component is equal to the variance of the chi-squared approximation plus further smaller terms. These terms are found to be negative and heavily dependent on the size of k and N. Higher terms may only be neglected when k and N are relatively large. Further proof of this is discussed in Section 9.5.

9.4.2 Variance of $\left[N - \sum_{j=1}^{q} \left[\frac{R^2_{.j}}{N_{.j}}\right]\right]$

Following the same procedure as Section 9.4.1, let q be the number of samples and c the number of observations within each sample and of equal size

$$\mathrm{Var}\left[N - \sum_{j=1}^{q} \left[\frac{R^2_{.j}}{N_{.j}}\right]\right] = -q - \rho^2[2(c-1)q] - \rho^4[q(c-1)^2]$$

$$+ \frac{q}{c^2}\left[\frac{c!}{(c-4)!}\rho^4 + \frac{2c!}{(c-3)!}\rho^2(2 + \rho_2)\right.$$

$$\left. + \frac{c!}{(c-2)!}(2 + 4\rho^2 + \rho_2^2) + c\right] \tag{9.4.8}$$

Using the series equations of (9.4.5), for large k

$$\mathrm{Var}\left[N - \sum_{j=1}^{q} \left[\frac{R^2_{.j}}{N_{.j}}\right]\right] = \frac{1}{k^2}(2qc - 2q) + \frac{1}{k^3}\left[-qc + \frac{q}{c}\right]$$

$$+ \frac{1}{k^4}\left[\frac{-5qc}{32} + \frac{89q}{32} - \frac{53q}{32c}\right] + \frac{1}{k^5}\left[\frac{-5qc}{16} + \frac{7q}{8} - \frac{9q}{16c}\right]$$

$$+ \frac{1}{k^6}\left[\frac{9qc}{32} + \frac{7q}{32} - \frac{1q}{16c}\right] + 0\left[\frac{1}{k^7}\right] \tag{9.4.9}$$

Therefore

$$\mathrm{Var}\left\{k\left[N - \sum_{j=1}^{q} \left[\frac{R^2_{.j}}{N_{.j}}\right]\right]\right\} = 2q(c-1) + \frac{1}{k}\left[-qc + \frac{q}{c}\right] + \ldots\ldots \tag{9.4.10}$$

As with the total component of variation the variance of the residual component is equal to the variance of the chi-squared approximation plus smaller negative terms dependent on the size of k and N.

9.4.3 Variance of $\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2}{N}\right]$

Let q be the number of samples and c be the number of observations within each sample of equal size

$$\mathrm{Var}\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2}{N}\right] = \frac{1}{(qc)}\left[\frac{(qc)!}{((cq) - 4)!}\rho^4\right.$$

$$\left. + \frac{2(qc)!}{((qc) - 3)!}\rho^2(2 + \rho_2) + \frac{(qc)!}{((qc) - 2)!}(2 + 4\rho^2 + \rho_2^2) + c\right]$$

$$- \frac{q}{c^2}\left[\frac{c!}{(c - 4)!}\rho^4 + \frac{2c!}{(c - 3)!}\rho^2(2 + \rho_2) + \right.$$

$$\left. + \frac{c!}{(c - 2)!}(2 + 4\rho^2 + \rho_2^2) + c\right]$$

$$+ (q - 1) - 2(q - 1)\rho^2 + \rho^4(qc^2 - (qc)^2 + q - 1) \tag{9.4.11}$$

Using the series equations of (9.4.5), for large k

$$\mathrm{Var}\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2}{N}\right] = \frac{1}{k^2}(2(q - 1)) + \frac{1}{k^3}\left[\frac{-1}{c}\left[q - \frac{1}{q}\right]\right]$$

$$+ \frac{1}{k^4}\left[\frac{-53}{32qc} + \frac{53q}{32c} - \frac{89q}{32} + \frac{89}{32}\right]$$

$$+ \frac{1}{k^5}\left[\frac{-9}{16qc} + \frac{9q}{16c} - \frac{7q}{8} + \frac{7}{8}\right]$$

$$+ 0\left[\frac{1}{k^6}\right] \tag{9.4.12}$$

Therefore

$$\operatorname{Var}\left\{k\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right]-\frac{R^2}{N}\right]\right\} = 2(q-1)+\frac{1}{k}\left[\frac{-1}{c}\left[q-\frac{1}{q}\right]\right]+\ldots\ldots \qquad (9.4.13)$$

Again it is important to note that the variance of the between component is equal to the variance of the chi-squared approximation plus smaller further terms. However, Section 9.5 shows how the size of k and N within the components have substantial effect on the size and accuracy of the variance, particularly so on the between measure of variation.

9.5 The Adequacy of the New Procedure, for Large k

As we have discussed in Chapter 4 and showed in Chapter 6, the approach by Watson and Williams, and adapted by Stephens, is the principal approach for testing differences between mean directions of different samples with assumed equal concentration parameter. The procedure developed in Chapter 8 is ultimately designed to analyse larger experimental situations where extensions to Watson and Williams have been shown to breakdown. However, it is necessary to compare the accuracy and power of the new test statistics with the alternative tests for the one-way classification before investigating its suitability for larger designs.

As in Chapter 5, for testing the adequacy of the extended techniques from Watson and Williams, simulation techniques have been used to examine the accuracy of the approximations to the distributions of the components of (8.3.3). The observations from the von Mises distribution specified by the null hypothesis were generated by the computer method outlined in Appendix B. 10,000 sets of samples of various size were drawn from the von Mises distribution with k = 2, 3, 4, 5 and 10. For the sake of conformity and in order to further check that the simulation techniques being used were satisfactory, the same sample sizes are analysed as in Upton (1974). Five multi-sample experimental designs are examined varying in size, with N ranging from 10 to 60.

To verify the findings of Section 9.4, the first two moments of the components for total and residual measures of variation are given in Table 9.5.1. The mean of the chi-squared approximation value for each component is seen to be an excellent fit increasing in accuracy as k increases. If k is unknown and the improvement factor (9.2.5) is used the accuracy of the chi-squared approximation mean value is maintained. As shown in Section 9.4 the variance of each component is seen to be below its chi-squared approximation value and dependent on the size of N and k. As N and predominantly k increase, the accuracy of the variance increases.

This deficiency within both components is reflected in Table 9.5.2 and 9.5.3, showing the accuracy of the upper percentage points to the χ^2 distribution with associated degrees of freedom. It is clearly seen that α_i, the simulated proportion of the components, approaches α, the χ^2 theoretical or significance level, as N and k increase. Similarly, due to the size of the chi-squared approximation variance values the probability of accepting the null hypothesis when in fact it is false, a type II error, is increased; and unacceptedly so for k=2 and small N. Tables 9.5.2 and 9.5.3 also give the comparable accuracy of the Watson and Williams component improved by Stephens. For increased understanding the same 10,000 sets of observations for each set of samples were used. It is seen for both components that while the new procedure components overestimate the theoretical proportion, following Stephens improvement Watson and Williams components slightly underestimate the theoretical proportion or significance level of the associated χ^2.

TABLE 9.5.1 Comparing the mean and variance of the χ^2 approximations for the total and residual measures of variation against their expected values

SAMPLE SIZES	k	$K(N-R^2_{..}/N)$ Mean	(Variance)	SAMPLE SIZES	$K(N-\Sigma(R^2_{.j}/N_{.j})$ Mean	(Variance)
5,5	2	9.198	(11.759)	5,5	8.182	(10.338)
	3	9.161	(15.43)		8.193	(13.481)
Expected	4	9.158	(15.752)	Expected	8.116	(13.706)
Mean = 9	5	8.913	(15.838)	Mean = 8	7.916	(14.701)
Variance = 18	10	9.063	(17.979)	Variance = 16	8.03	(15.872)
10,10	2	19.472	(24.262)	10,10	18.488	(22.902)
	3	19.48	(33.541)		18.454	(31,586)
Expected	4	19.214	(34.528)	Expected	18.2	(32.139)
Mean = 19	5	19.278	(35.759)	Mean = 18	18.185	(33.451)
Variance = 38	10	19.057	(36.987)	Variance = 36	18.078	(34.426)
15,35	2	50.21	(66.342)	15,35	49.073	(64.689)
	3	50.184	(86.082)		48.896	(84.063)
Expected	4	49.82	(89.812)	Expected	48.742	(87.554)
Mean = 49	5	49.419	(93.547)	Mean = 48	48.47	(91.426)
Variance = 98	10	49.043	(95.72)	Variance = 96	48.038	(93.122)

TABLE 9.5.1 continued:-

SAMPLE SIZES	k	$K(N-R^2_{i.}/N)$ Mean	(Variance)	SAMPLE SIZES	$K(N-\Sigma(R^2_{.j}/N_{.j}))$ Mean	(Variance)
20,20	2	40.068	(50.524)	20,20	39.051	(48.999)
	3	39.982	(67.987)		39.08	(65.935)
Expected	4	39.505	(72.355)	Expected	38.503	(69.652)
Mean = 39	5	39.385	(73.141)	Mean = 38	38.281	(70.664)
Variance = 78	10	39.086	(73.167)	Variance = 76	38.1	(71.761)
10,20,30	2	60.307	(75.128)	10,20,30	58.461	(71.957)
	3	60.419	(101.288)		58.412	(97.299)
Expected	4	60.041	(108.249)	Expected	58.025	(103.984)
Mean = 59	5	59.628	(114.688)	Mean = 57	57.487	(109.746)
Variance = 118	10	59.144	(115.848)	Variance = 114	57.148	(110.968)
15,15,15,15	2	60.54	(79.436)	15,15,15,15	57.269	(75.923)
	3	60.443	(101.921)		57.274	(95.479)
Expected	4	60.084	(106.933)	Expected	56.987	(100.86)
Mean = 59	5	59.628	(113.928)	Mean = 56	56.487	(108.219)
Variance = 118	10	59.153	(114.521)	Variance = 112	56.153	(110.321)

TABLE 9.5.2 Comparing the accuracy of the two χ^2 approximations with N-1 degrees of freedom representing the total measure of variation

SAMPLE SIZES	K	$K(N-R_{i.}^2/N)$			$2Y(N-R_{..})$		
		α			α		
		0.90	0.95	0.99	0.90	0.95	0.99
5,5	2	0.9308	0.9852	0.999	0.8871	0.9462	0.9937
	3	.9074	.9616	.9974	.8904	.944	.9874
	4	.9041	.9587	.9949	.8979	.9491	.991
	5	.9055	.9581	.9943	.9043	.9526	.9909
	10	.9001	.9515	.9898	.8986	.9495	.9874
10,10	2	0.9291	0.9841	0.9993	0.8931	0.9488	0.9932
	3	.8979	.9549	.9948	.8836	.9387	.9861
	4	.9065	.9577	.9923	.9017	.9481	.9883
	5	.8969	.9515	.9915	.8943	.9456	.9888
	10	.9011	.9509	.9939	.9019	.95	.9936
15,35	2	0.9232	0.9737	0.9989	0.8792	0.9463	0.991
	3	.8965	.9545	.9921	.8877	.9445	.9876
	4	.8925	.9511	.993	.8931	.9482	.991
	5	.90	.9524	.9923	.9042	.951	.991
	10	.9027	.9543	.9903	.9063	.9556	.9895

The table gives the proportion of 10,000 Monte Carlo samples for which the two approximations fell below the χ^2_{N-1} value fpr which the theoretical proportion should be α.

TABLE 9.5.2 continued:-

SAMPLE SIZES	k	$K(N-R^2_{\cdot\cdot}/N)$ α 0.90	0.95	0.99	$2Y(N-R_{\cdot\cdot})$ α 0.90	0.95	0.99
	2	0.9287	0.9748	0.9983	0.8937	0.9471	0.9907
	3	.8945	.9535	.992	.8858	.9404	.9925
20,20	4	.8981	.9517	.9916	.8968	.9477	.9891
	5	.8994	.9525	.9923	.9027	.951	.9905
	10	.9043	.9563	.9929	.908	.9583	.9922
	2	0.9219	0.9731	0.9979	0.89	0.9477	0.9917
	3	.8893	.9488	.9919	.8856	.9399	.9878
10,20,30	4	.889	.9501	.9933	.8916	.9476	.9905
	5	.896	.9491	.9902	.9007	.9494	.9885
	10	.9025	.9573	.991	.9075	.9572	.9909
	2	0.9221	0.9726	0.998	0.8846	0.9414	0.9871
	3	.8892	.9489	.9921	.8857	.9398	.9879
15,15,15,15,	4	.90	.9515	.9917	.9034	.949	.99
	5	.896	.9495	.99	.9005	.9494	.9885
	10	.9055	.9527	.9898	.9128	.9542	.9896

TABLE 9.5.3 Comparing the accuracy of the two χ^2 approximations with N-q degrees of freedom representing the residual measure of variation

SAMPLE SIZES	k	$K(N-\Sigma(R^2_{.j}/N_{.j}))$ α 0.90	0.95	0.99	$2Y(N-\Sigma R_{.j})$ α 0.90	0.95	0.99
5,5	2	0.9321	0.98	0.9988	0.8849	0.9454	0.9927
	3	.9115	.9627	.9971	.8848	.9423	.984
	4	.9061	.9605	.9962	.8937	.9448	.9906
	5	.9142	.9602	.9947	.9067	.9514	.9913
	10	.9008	.9508	.9898	.8983	.9475	.9888
10,10	2	0.9348	0.9812	0.9989	0.8856	0.9482	0.9925
	3	.8972	.953	.9952	.8756	.9365	.9846
	4	.9076	.9567	.9934	.8984	.9471	.99
	5	.8986	.9528	.993	.8926	.9466	.9886
	10	.9023	.9555	.9916	.8974	.9497	.9933
15,35	2	0.9236	0.9742	0.9979	0.8772	0.9434	0.9906
	3	.8952	.9545	.992	.8847	.9678	.9873
	4	.8944	.9514	.9934	.8929	.947	.9896
	5	.8996	.9514	.9923	.9009	.9498	.9907
	10	.9023	.9553	.991	.9063	.9568	.9908

The table gives the proportion of 10,000 Monte Carlo samples for which the two approximations fell below the χ^2_{N-q} value for which the theoretical proportion should be α.

TABLE 9.5.3 continued:-

SAMPLE SIZES	k	$K(N-\Sigma(R^2_{.j}/N_{.j}))$ α 0.90	0.95	0.99	$2Y(N-\Sigma R_{.j})$ α 0.90	0.95	0.99
	2	0.933	0.9771	0.9987	0.8882	0.9458	0.9909
	3	.895	.9538	.9924	.8851	.9379	.9857
20,20	4	.9027	.9529	.993	.8992	.9465	.9877
	5	.901	.9555	.9933	.9007	.9517	.9915
	10	.9067	.956	.9931	.9093	.9573	.9924
	2	0.9307	0.9768	0.9989	0.8819	0.946	0.9903
	3	.8884	.9506	.9923	.8794	.9387	.987
10,20,30	4	.8918	.95	.9925	.89	.9456	.9902
	5	.8961	.9493	.991	.899	.9483	.989
	10	.9002	.953	.9916	.904	.9537	.9916
	2	0.9231	0.97	0.9981	0.8792	0.937	0.9868
	3	.8858	.951	.993	.8729	.937	.9861
15,15,15,15	4	.8978	.9552	.9925	.8968	.9502	.989
	5	.8952	.9516	.9914	.897	.9491	.9883
	10	.9036	.9548	.99	.907	.9556	.9896

Table 9.5.4 gives the first two moments for the between measure of variation component. Once again the mean value of the component chi-squared approximation is a very good fit increasing in accuracy as k increases. The deficiency in the accuracy of the chi-squared approximation variance is again shown. This is slightly worse than for the total and residual components.

Following examination of the fitted components percentiles to the upper 50% of the theoretical χ^2 distribution, several improvement factors were tested to increase the accuracy of the chi-squared approximation variance without greatly impairing the accuracy of the mean. The improvement factor β, (9.5.1), was chosen as a balance between the two moments, acceptable for all large k

$$\frac{1}{\beta} = 1 - \frac{1}{5\hat{k}} - \frac{1}{10\hat{k}^2} \qquad (9.5.1)$$

Table 9.5.4 gives the comparable moments for each set of sample sizes multiplying k by β and shows how the mean of the chi-squared approximation has increased slightly from its desired value whilst the associated variance has improved appreciably.

Table 9.5.5 gives the accuracy of the improved between measure of variation to the upper percentage points of the associated χ^2 distribution with comparable statistics for Stephens measure. By introducing β the accuracy of α_i the simulated proportion, is greatly improved, increasing in accuracy as k increases.

It is important to note that to acquire this accuracy with the component distribution variances being below the desired theoretical values implies that the accuracy of the lower percentage points have deteriorated. However, the accuracy of the whole component distribution is still very good, although Stephens measure has the superior fit across all percentage points.

TABLE 9.5.4 Comparing the mean and variance of the χ^2 approximations for the between measure of variation against its expected values

SAMPLE SIZES	k	$K(\Sigma(R^2_{.j}/N_{.j})-R^2_{..}/N)$ Mean	(Variance)	$KB(\Sigma(R^2_{.j}/N_{.j})-R^2_{..}/N)$ Mean	(Variance)
5,5	2	1.131	(1.148)	1.174	(1.597)
	3	1.042	(1.329)	1.14	(1.706)
Expected	4	0.94	(1.481)	1.098	(1.735)
Mean = 1	5	0.995	(1.70)	1.042	(1.73)
Variance = 2	10	1.092	(1.863)	1.032	(1.927)
10,10	2	1.097	(1.104)	1.162	(1.485)
	3	1.086	(1.401)	1.117	(1.679)
Expected	4	1.104	(1.475)	1.094	(1.846)
Mean = 1	5	1.003	(1.698)	1.081	(1.815)
Variance = 2	10	0.979	(1.871)	1.009	(1.792)
15,35	2	1.077	(1.098)	1.176	(1.594)
	3	1.101	(1.33)	1.101	(1.602)
Expected	4	1.04	(1.464)	1.087	(1.944)
Mean = 1	5	0.987	(1.72)	0.997	(1.762)
Variance = 2	10	0.995	(1.876)	1.021	(1.923)

TABLE 9.5.4 continued:-

SAMPLE SIZES	K	$K(\Sigma(R^2_{.j}/N_{.j})-R^2_{..}/N)$ Mean	(Variance)	$KB(\Sigma(R^2_{.j}/N_{.j})-R^2_{..}/N)$ Mean	(Variance)
20,20	2	1.108	(1.113)	1.163	(1.529)
	3	1.071	(1.372)	1.108	(1.629)
Expected	4	1.024	(1.48)	1.112	(1.903)
Mean = 1	5	0.997	(1.723)	1.071	(1.888)
Variance = 2	10	1.002	(1.894)	1.006	(1.919)
10,20,30	2	2.142	(2.274)	2.281	(3.217)
	3	2.097	(2.639)	2.24	(3.393)
Expected	4	1.989	(2.972)	2.136	(3.447)
Mean = 2	5	2.007	(3.541)	2.127	(3.588)
Variance = 4	10	2.019	(3.647)	2.051	(3.8)
15,15,15,15	2	3.232	(3.495)	3.48	(4.89)
	3	3.27	(3.959)	3.276	(5.035)
Expected	4	3.091	(4.477)	3.181	(5.228)
Mean = 3	5	3.048	(5.291)	3.157	(5.356)
Variance = 6	10	3.074	(5.468)	3.045	(5.678)

TABLE 9.5.5 Comparing the accuracy of the two χ^2 approximations with q-1 degrees of freedom representing the between measure of variation

SAMPLE SIZES	k	$KB(\Sigma(R^2_{.j}/N_{.j})-R^2_{..}/N)$			$2k(\Sigma R_{.j}-R_{..})$		
		α			α		
		0.90	0.95	0.99	0.90	0.95	0.99
5,5	2	0.8909	0.9561	0.9939	0.8881	0.9435	0.9862
	3	.8992	.9526	.9923	.8963	.946	.9876
	4	.8992	.9545	.9925	.8931	.9496	.9888
	5	.9042	.9546	.9939	.9022	.9493	.991
	10	.8974	.9459	.9912	.8942	.9447	.9898
10,10	2	0.897	0.9569	0.9949	0.8934	0.9435	0.9873
	3	.8978	.9547	.9952	.9002	.948	.9875
	4	.891	.9488	.9913	.8894	.9419	.989
	5	.8943	.9497	.9925	.8887	.945	.9907
	10	.8998	.955	.992	.8978	.9534	.9915
15,35	2	0.897	0.9552	0.9947	0.8981	0.9442	0.9891
	3	.8995	.957	.9949	.8978	.95	.9923
	4	.8998	.9531	.9901	.8987	.9491	.9869
	5	.9106	.9601	.9937	.911	.9577	.9925
	10	.90	.9517	.991	.8966	.95	.9899

The table gives the proportion of 10,000 Monte Carlo samples for which the two approximations fell below the χ^2_{q-1} value for which the theoretical proportion should be α.

TABLE 9.5.5 continued:-

SAMPLE SIZES	k	$\kappa B(\Sigma(R^2_{.j}/N_{.j})-R^2_{..}/N)$ 0.90	α 0.95	0.99	$2k(\Sigma R_{.j}-R_{..})$ 0.90	α 0.95	0.99
	2	0.8993	0.9569	0.9951	0.9016	0.9485	0.9887
	3	.9041	.9568	.9923	.9015	.9509	.9902
20,20	4	.8979	.9511	.99	.8928	.9468	.9879
	5	.8945	.9493	.9913	.89	.9463	.9896
	10	.901	.95	.9905	.9002	.9482	.9892
	2	0.8993	0.9567	0.9943	0.8983	0.9489	0.9891
	3	.8995	.9488	.9923	.9002	.95	.9895
10,20,30	4	.9051	.9539	.992	.9031	.9521	.9898
	5	.903	.9519	.9906	.9011	.9489	.9898
	10	.9011	.9516	.99	.9006	.9503	.9891
	2	0.8917	0.9581	0.9922	0.8931	0.9505	0.989
	3	.8982	.951	.994	.903	.95	.9925
15,15,15,15	4	.8985	.9529	.9926	.8977	.9493	.9895
	5	.898	.9547	.9922	.8959	.9519	.99
	10	.9013	.953	.9905	.9004	.9517	.9893

Following the introduction of β the test statistic components can now be expressed as;

$$k\left[N - \frac{R^2}{N}\right] = k\beta\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2}{N}\right] + k\left[N - \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right]\right] \tag{9.5.2}$$

The associated test statistic, Q_1, for the null hypothesis of no difference between q treatments or samples is given as;

$$Q_1 = \frac{k\beta(N - q)\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2}{N}\right]}{k(q - 1)\left[N - \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right]\right]} \tag{9.5.3}$$

which has an approximate F–distribution with (q–1) and (N–q) degrees of freedom.

This may now be examined and compared to the associated test statistics from Stephens (4.4.7), and Uptons G–test (4.4.8). (See Table 9.5.6.) Test statistic (9.5.3) and (4.4.7) are compared to the theoretical significance levels of an F–distribution, while (4.4.8) is compared to those of the χ^2 distribution.

Test statistic (9.5.3) is seen to be accurate for large concentration parameter and increasing slightly in accuracy as N increases. This dependence on the size of N is more pronounced for k equal to 2. As stated by Upton (1974) the G–test is greatly affected by the size of N and is inappropriate for small sample sizes. Stephens test statistic is again seen as the 'best fitting' statistic when examined across all large k and varying sizes of N.

TABLE 9.5.6 Comparison of the three test statistics; Q_1 and Stephens compared to the F distribution and Uptons G-test compared to the χ^2 distribution

SAMPLE SIZES	k	Q_1 Equation (9.5.3)			STEPHENS Equation (4.4.7)			UPTONS G-TEST Equation (4.4.		
		α			α			α		
		0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
5,5	2	0.9131	0.962	0.9923	0.9003	0.951	0.9896	0.8671	0.9252	0.9797
	3	.9092	.9583	.991	.8994	.952	.9883	.868	.9271	.98
	4	.9037	.9561	.9914	.8959	.9497	.9905	.8645	.9243	.9828
	5	.905	.956	.994	.895	.9504	.9928	.8646	.9253	.983
	10	.8962	.9496	.9922	.8914	.9472	.9911	.8605	.9187	.9802
10,10	2	0.9107	0.9645	0.9943	0.9033	0.9506	0.9909	0.8902	0.9423	0.987
	3	.9102	.9587	.9921	.9037	.9532	.9894	.8923	.9447	.9865
	4	.9048	.9532	.9902	.8957	.9451	.9875	.8834	.9351	.985
	5	.907	.9531	.9924	.8985	.9473	.9909	.8863	.9382	.9847
	10	.9018	.9512	.9916	.898	.9488	.9921	.885	.9392	.9883
15, 35	2	0.909	0.9606	0.9941	0.9029	0.9505	0.9905	0.8982	0.947	0.9893
	3	.9087	.9601	.9946	.90	.9507	.9918	.8972	.9498	.9912
	4	.9051	.9545	.9904	.8986	.9477	.9882	.8966	.9457	.9871
	5	.914	.9595	.9931	.911	.9553	.993	.9085	.9542	.9918
	10	.902	.954	.9927	.8976	.9498	.9914	.893	.9472	.9911

TABLE 9.5.6 continued:-

SAMPLE SIZES	k	Q_1 Equation (9.5.3) α			STEPHENS Equation (4.4.7) α			UPTONS G-TEST Equation (α		
		0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
20,20	2	0.9071	0.9618	0.9946	0.9065	0.9538	0.9905	0.9017	0.9498	0.98
	3	.91	.9587	.9933	.906	.9522	.9894	.9016	.9491	.98
	4	.904	.955	.991	.894	.9504	.988	.891	.9472	.98
	5	.8986	.953	.9914	.8893	.9475	.9887	.8852	.944	.98
	10	.9044	.953	.99	.9014	.9486	.9893	.8972	.945	.98
10,20,30	2	0.9106	0.9617	0.9951	0.9064	0.9566	0.9927	0.9015	0.9542	0.99
	3	.911	.9583	.9928	.908	.9523	.9881	.9051	.9513	.98
	4	.9133	.959	.9937	.9061	.9518	.99	.9045	.9503	.98
	5	.9063	.9557	.9923	.899	.949	.9889	.8979	.9467	.98
	10	.90	.9544	.9909	.8971	.9518	.9905	.8925	.9477	.98
15,15,15,15	2	0.9066	0.9601	0.9931	0.9055	0.9506	0.991	0.8997	0.948	0.98
	3	.9092	.9611	.994	.9062	.9572	.9906	.9035	.9554	.99
	4	.9083	.9587	.9939	.9022	.9522	.9918	.8988	.9497	.99
	5	.9081	.9583	.993	.9004	.9523	.9903	.8959	.9497	.98
	10	.904	.9519	.9905	.9016	.9485	.99	.8955	.946	.98

9.6 Comparing the Power of the Tests, for Large k

In previous sections the components and test statistics introduced have been compared with each other principally on the basis of their apparent goodness of fit to their respective distributions. All these tests involve the use of many approximations in their construction;

a) The approximation of the ratio of Bessel functions in the likelihood ratio test as a short power series (Chapter 3.6).

b) The approximation to obtain an explicit value for the maximum likelihood estimate of the concentration parameter (Chapter 3).

c) The use of the asymptotic result about the distribution of $-2\log\lambda$ (Chapter 3.1).

d) Occasionally an approximation is introduced to simplify the test statistic.

e) The modification of the statistic by correcting its expected value (Chapter 2.5).

It was similarly noted by Upton (1970) that because of all these approximations it is often surprising that so many of the tests derived using this procedure are good fits to their respective χ^2 and F distributions. It is for these reasons that it is important to examine the statistics relative goodness of fit and establish their range of validity.

What is also important is to examine the relative power of the statistics against any available alternatives. In order to compare the test statistic (9.5.3) against alternative multi-sample tests given by Stephens (4.4.7) and Upton (4.4.8) it is unnecessary to conduct a complete investigation of all the different sample situations we have already used. Table 9.6.1 gives the relative power of the three tests by studying two sets of samples, with one sample mean direction set approximately 30° from the true value of the other samples. It is clear from the table that Uptons and Stephens tests are virtually identical in power, with the Q-test slightly less powerful improving in comparison as k increases. The power of all three tests naturally increases as the underlying distribution becomes more peaked.

As has previously been noted likelihood ratio theory produces tests that are asymptotically uniformly most powerful. Therefore the tests derived in this manner examining the required hypothesis will be at least as powerful as any other test. The test statistic (9.5.3) initially constructed by likelihood methods but requiring a further test to eliminate an assumption of unequal concentration parameter would not, expectedly, be quite as powerful as that derived without this restriction. Nevertheless this loss of power is not very large and is compensated by its increased range of application as a generalised approach for larger experimental designs.

TABLE 9.6.1 Comparison of the power of three alternative tests, for large concentration parameter

TEST	$N_1 = N_2 = 20$, $\theta_1 - 30^\circ = \theta_2$ 10%	5%	1%	$N_1 = 10$ $N_2 = 20$ $N_3 = 30$, $\theta_1 - 30^\circ = \theta_2 = \theta_3$ 10%	5%	1%
k = 2						
UPTONS' G-TEST (4.4.8)	0.6031	0.4711	0.2432	0.4468	0.3284	0.1404
STEPHENS TEST (4.4.7)	.604	.4704	.2409	.44	.3207	.1384
Q-TEST (9.5.3)	.5793	.4455	.1913	.4088	.2795	.1037
k = 4						
UPTONS' G-TEST	0.913	0.8479	0.6309	0.7977	0.6925	0.4334
STEPHENS' TEST	.9129	.8492	.6315	.7961	.6885	.428
Q-TEST	.9004	.8248	.6024	.7849	.6703	.4081
k = 6						
UPTONS' G-TEST	0.982	0.9634	0.8738	0.9436	0.8866	0.7002
STEPHENS TEST	.9824	.963	.8725	.9427	.8806	.6938
Q-TEST	.9789	.9574	.8545	.9392	.8203	.6684

The table gives the proportion of 10,000 Monte Carlo samples which exceeded the upper stated percentage points of the relevant distribution.

9.7 The Adequacy of the New Procedure, for Small k

As previously stated, the new procedure for small k is based on the likelihood ratio test for the null hypothesis $\beta_1 = \ldots = \beta_q$, $k_1 = \ldots = k_q$ against its general alternative hypothesis. For one-way analysis problems Mardia's likelihood ratio test (4.4.3) for the null hypothesis $\beta_1 = \ldots = \beta_q$ with assumed equal k will be asymptotically uniformly most powerful as a direct test of the mean directions. As with test statistic (4.4.3) the comparable test statistic (9.2.6) only involves the main factor effect and cannot take account of the total or more importantly the residual or error effect. However, unlike (4.4.3) the loss of power of test statistic (9.2.6) may be compensated by its possible application in larger experimental situations to be discussed in Chapter 10. Table 9.7.1 compares the accuracy of the two χ^2 approximations (4.4.3) and (9.2.6) following 10,000 simulations of varying concentration parameter and sample size. The sample sizes range from 10 to 60 using the same sample sets as for large k. Before the tests are applied the equality of concentration parameter is checked using the appropriate tests from Section 4.5. Ranges of small k are given as its accuracy cannot be assured for such small values. From Table 9.7.1, test statistic (9.2.6) is seen to be less susceptible to smaller sample sizes than (4.4.3). As the sample sizes increase the accuracy of the two tests become more comparable with both showing good fits to the upper percentage points of the χ^2 distribution with associated degrees of freedom. For both tests α_i, the simulated proportion of the component, approaches α, the χ^2 theoretical proportion or significance, as N and k increases.

TABLE 9.7.1 Comparing the accuracy of test statistics (9.2.6) and (4.4.3) as χ^2 approximations with 2(q-1) degrees of freedom

SAMPLE SIZES	k	TEST STATISTIC (9.2.6) α			TEST STATISTIC (4.4.3) α		
		0.90	0.95	0.99	0.90	0.95	0.99
5,5	0-0.5	0.9102	0.9621	0.994	0.9727	0.9945	0.9992
	0.5-1.0	.898	.9546	.994	.8717	.9394	.993
10,10	0-0.5	0.9113	0.9591	0.9907	0.944	0.98	0.9983
	0.5-1.0	.9039	.954	.9923	.8805	.9406	.9885
15,35	0-0.5	0.894	0.9482	0.9892	0.8982	0.9528	0.9942
	0.5-1.0	.897	.9485	.9906	.889	.9417	.9879
20,20	0-0.5	0.9044	0.9541	0.9917	0.9034	0.9575	0.9947
	0.5-1.0	.8988	.9465	.9913	.8869	.9433	.9878
10,20,30	0-0.5	0.9058	0.9556	0.9936	0.9182	0.9613	0.9951
	0.5-1.0	.9038	.952	.9911	.8793	.9401	.9865
15,15,15,15	0-0.5	0.9061	0.9548	0.992	0.9202	0.9624	0.9962
	0.5-1.0	.9074	.9481	.9927	.8787	.9484	.987

Table 9.7.2 takes the tests statistic (9.2.6) out of its proven range and compares it to equivalent tests in the concentration parameter range 1 to 2. The comparable tests are those stated by Stephens and Upton given in Section 4.4, and used previously when examining tests for large k. Stephens test is shown to be suitable for k as small as 1, while Upton's test is suitable for R/N as small as 0.6 ($k \sim 1.5$). Table 9.7.2 shows test statistic (9.2.6) to be comparable to the equivalent tests, whilst all three tests show good fits to the upper percentage points. Once again it is important to note that the equality of concentration parameters has been tested prior to examination of mean directions.

TABLE 9.7.2 Comparison of the three test statistics; (9.2.6) and Uptons G-test compared to the χ^2 distribution and Stephens compared to the F distribution

SAMPLE SIZES	K	Test Statistic (9.2.6) α			STEPHENS Equation (4.4.7) α			UPTONS G-test Equation (4.4.8) α		
		0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
5, 5	1.25	0.8859	0.9454	0.9917	0.8885	0.9464	0.992			
	1.75	.8984	.945	.988	.9048	.9533	.9922	0.8668	0.9225	0.9808
10,10	1.25	0.8971	0.9523	0.9922	0.8968	0.9513	0.9914			
	1.75	.8929	.9461	.9837	.8991	.9486	.9874	0.8858	0.9375	0.9837
15,35	1.25	0.8891	0.9411	0.9874	0.899	0.9456	0.9886			
	1.74	.8932	.9455	.988	.8941	.9463	.989	0.8902	0.9413	0.9878
20,20	1.25	0.897	0.9502	0.9885	0.8994	0.9507	0.9892			
	1.75	.8991	.9473	.9894	.9024	.9536	.9912	0.8962	0.9481	0.9903
10,20,30	1.25	0.9041	0.951	0.9871	0.8973	0.9483	0.9881			
	1.75	.8899	.9445	.986	.8966	.9489	.9888	0.8923	0.943	0.9871
15,15,15,15	1.25	0.8991	0.95	0.9915	0.8978	0.9513	0.991			
	1.75	.8924	.9402	.9863	.8997	.95	.9907	0.8917	0.9445	0.9886

9.8 Comparing the Power of the Tests, for Small k

As for large k, it is important to examine the relative power of the test statistic against any available alternatives. To compare test statistic (9.2.6) against Mardia's test statistic (4.4.3) for $k < 1$, and against Stephens (4.4.7) and Uptons (4.4.8) for $k > 1$ it is unnecessary to investigate all the different sample situations of Section 9.7. Tables 9.8.1 and 9.8.2 show the relative power of the four tests by studying two sets of samples, one with equal sample sizes of 20, the other with differing sample sizes of 10, 20 and 30. Table 9.8.1 examines the power of the test with one sample mean direction set approximately 30° from the true value of the other samples, as in the power examination for large k. Table 9.8.2, however, gives a further test when one sample mean direction is set 90° from the other samples. A larger displacement has been used for the examination of small k, since to acquire a significant difference between samples when simulating almost randomness is difficult and larger fluctuations in test statistic would be expected. In both tables it is clearly seen how the detection of a significant difference is increased as k increases. When a displacement of 90° is used with $k = 1.75$ all three test statistics indicate the presence of a displacement on almost 100 percent of occasions.

Examining the power between the tests for $k < 1$ shows Mardia's statistic (4.4.3) to be slightly more powerful than test statistic (9.2.6) in both situations and across all percentage points. For k between 1 and 2 Stephens test statistic dominates close to 1 while all three tests appear to be identical in power for k increasing towards 2.

Test statistic (9.2.6), constructed using maximum likelihood techniques but requiring examination of the sample concentration parameter, is seen to be almost as powerful as test statistics (4.4.3) and (4.4.7) but with the possible added application to larger experimental situations.

TABLE 9.8.1 <u>Comparison of the power of three alternative tests, for small concentration parameter</u>

TEST	$N_1 = N_2 = 20$, $\theta_1 - 30° = \theta_2$			$N_1=10$ $N_2=20$ $N_3=30$, $\theta_1 - 30° = \theta_2 = \theta_3$		
	10%	5%	1%	10%	5%	1%
$0 < k < 0.5$						
TEST STATISTIC (4.4.3)	0.1181	0.0612	0.0078	0.0872	0.0322	0.0057
TEST STATISTIC (9.2.6)	.1207	.0601	.0104	.1055	.0505	.0105
$0.5 \leqslant k \leqslant 1.0$						
TEST STATISTIC (4.4.3)	0.2325	0.1474	0.0462	0.1854	0.1091	0.0275
TEST STATISTIC (9.2.6)	.2046	.1263	.0375	.151	.0859	.0187
$k = 1.25$						
STEPHENS TEST (4.4.7)	0.3627	0.2493	0.0938	0.2751	0.1743	0.0538
TEST STATISTIC (9.2.6)	.3381	.2385	.0915	.2499	.1543	.0525
$k = 1.75$						
STEPHENS' TEST (4.4.7)	0.534	0.4057	0.1925	0.4053	0.2818	0.1199
UPTONS' TEST (4.4.8)	.5504	.4301	.2164	.3907	.2683	.1082
TEST STATISTIC (9.2.6)	.522	.4116	.2175	.3845	.2728	.1201

TABLE 9.8.2 Comparison of the power of three alternative tests, for small concentration parameter

TEST	$N_1 \pm N_2 = 20$, $\theta_1 - 90° = \theta_2$			$N_1=10$ $N_2=20$ $N_3=30$, $\theta_1 - 90° = \theta_2 = \theta_3$		
	10%	5%	1%	10%	5%	1%
$0 < k < 0.5$						
TEST STATISTIC (4.4.3)	0.1781	0.0916	0.0161	0.1233	0.0541	0.0065
TEST STATISTIC (9.2.6)	.1682	.0994	.0232	.1485	.0836	.0158
$0.5 < k < 1.0$						
TEST STATISTIC (4.4.3)	0.7485	0.6351	0.4056	0.5953	0.4638	0.2185
TEST STATISTIC (9.2.6)	.6912	.5925	.366	.5406	.388	.1661
$k = 1.25$						
STEPHENS TEST (4.4.7)	0.9807	0.9617	0.8771	0.9211	0.8671	0.695
TEST STATISTIC (9.2.6)	.9756	.9502	.8542	.9089	.8528	.6699
$k = 1.75$						
STEPHENS TEST (4.4.7)	0.9985	0.9957	0.9827	0.9921	0.981	0.9298
UPTONS TEST (4.4.8)	.999	.997	.988	.9932	.9837	.9831
TEST STATISTIC (9.2.6)	.9989	.9969	.9881	.9936	.9853	.954

9.9 Summary

As the exact distribution of the new test statistic components have been seen to be intractable by an exact theoretical approach for both large and small k, other methods to examine their distribution have been used. A comprehensive step approach was initiated for both situations of large or small concentration parameter

(1) to test the components validity by use of approximations in a theoretical manner

(2) to obtain the expectation and variance of each chi-squared approximation component via Bessel function approximations

(3) simulation testing of the components to examine their accuracy to the upper percentage points of their associated chi-squared approximation and obtain and examine their first two moments

(4) to compare the new F or chi-squared test statistic against available alternatives, and finally

(5) to examine the power and robustness of the test statistics against these alternatives

Following the production of the first two moments, via the Bessel function approximation and simulation, an improvement factor β was derived to increase the accuracy of the distribution approximations for large k and therefore improve the final test statistic. The comparison of the new test statistic (9.5.3) to associated one-way classification tests was favourable although as it requires the prior testing of sample concentration parameters it is not a uniformly most powerful test. Nevertheless the new approach lends itself to an increased range of applications unlike the alternative techniques.

For small k only the associated component chi–squared approximation can be used as a test of difference since the corresponding total and residual components for small concentration parameter do not form chi–squared distributions. The test statistic chi–squared (9.2.6) also compared favourably to the available alternatives for small k.

As for large k, the tests of (4.4.3) and (4.4.7) are uniformly most powerful although generalisations of these tests to alternative experimental designs is not feasible.

CHAPTER 10

THE RANDOMISED COMPLETE BLOCK AND TWO-WAY DESIGNS VIA THE NEW APPROACH

10.1 Introduction

A comprehensive analysis of the distribution functions for the generalised approach has been carried out in Chapter 9. Here the new test statistics for the randomised complete block and two-way classification designs, together with their associated improvement factors, are produced. The relevant component statistics and test statistics are compared with their respective chi-squared and F distributions to ascertain their reliability and robustness for larger designs. In order not to repeat this analysis for every possibility the randomised complete block and two-way classification designs are used to represent and examine the adequacy of other larger more complex design situations. Although the components and test statistics are not investigated here, Chapter 11 extends the approach and analyses experimental situations using such methods as Latin-square and Split-plot designs.

10.2 Randomised Complete Block and Two-way Classification with Interaction Designs, for Large k

In Section 8.5 the randomised complete block design for the new approach was constructed via vector analysis and was shown to produce zero cross product terms. We may now test for any possible block effect, i, as well as any treatment effect, j, to produce the randomised complete block expression;

$$k\left[N - \frac{R^2_{..}}{N}\right] = k\left[\sum_{i=1}^{p}\left[\frac{R^2_{i.}}{N_{i.}}\right] - \frac{R^2_{..}}{N}\right] + k\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2_{..}}{N}\right]$$

$$+ k\left[N - \sum_{i=1}^{p}\left[\frac{R^2_{i.}}{N_{i.}}\right] - \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] + \frac{R^2_{..}}{N}\right] \qquad (10.2.1)$$

with associated independent chi–squared distributions

$$\chi^2_{(N-1)} = \chi^2_{(p-1)} + \chi^2_{(q-1)} + \chi^2_{(p-1)(q-1)}$$

As noted in Section 8.5 the first term on the right hand side of the expression is the measure of variation due to p treatments, the second term of similar form being a measure of variation due to q blocks. The final term represents the residual variation, where we assume that the experimental errors are independent and von Mises distributed.

Equation (10.2.1) produces the test statistic Z_6 to examine the null hypothesis that there is no difference between the p treatments

$$Z_6 = \frac{(p-1)(q-1)\left[\sum_{i=1}^{p}\left[\frac{R^2_{i.}}{N_{i.}}\right] - \frac{R^2_{..}}{N}\right]}{(p-1)\left[N - \sum_{i=1}^{p}\left[\frac{R^2_{i.}}{N_{i.}}\right] - \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] + \frac{R^2_{..}}{N}\right]} \qquad (10.2.2)$$

which has an F distribution with (p–1) and (p–1)(q–1) degrees of freedom. Similarly, the test statistic Z_7 is produced to test the null hypothesis that there is no difference between the q blocks

$$Z_7 = \frac{(p-1)(q-1)\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2_{..}}{N}\right]}{(q-1)\left[N - \sum_{i=1}^{p}\left[\frac{R^2_{i.}}{N_{i.}}\right] - \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] + \frac{R^2_{..}}{N}\right]} \qquad (10.2.3)$$

which has an F distribution with (q−1) and (p−1)(q−1) degrees of freedom.

Test statistics Z_6 and Z_7 indicate whether or not the mean directions are equal, they <u>do not</u> allow for discrimination amongst single mean directions which we were able to derive for the one-way analysis in Chapter 5.

Prior to examining the accuracy of the component chi-squared approximations of (10.2.1) and the associated F approximations Z_6 and Z_7, the test statistics for the two-way analysis with interaction may be constructed.

Using the same approach as above, the two-way classification with interaction may be seen as;

$$k\left[N - \frac{R^2_{...}}{N}\right] = k\left[\sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i.}}\right] - \frac{R^2_{...}}{N}\right] + k\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j.}}{N_{.j.}}\right] - \frac{R^2_{...}}{N}\right]$$

$$+ k\left[N - \sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R^2_{ij.}}{N_{ij.}}\right]\right]$$

$$+ k\left[\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R^2_{ij.}}{N_{ij.}}\right] - \sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i..}}\right] - \sum_{j=1}^{q}\left[\frac{R^2_{.j.}}{N_{.j.}}\right] + \frac{R^2_{...}}{N}\right]$$

(10.2.4)

with associated independent chi-squared distributions

$$\chi^2_{N-1} = \chi^2_{(p-1)} + \chi^2_{(q-1)} + \chi^2_{pq(m-1)} + \chi^2_{(p-1)(q-1)}$$

where m represents the number of observations within each cell. F test statistics are built as for the randomised complete block design;

Testing for differences between rows effects (i=1,2,....,p)

$$Z_8 = \frac{pq(m-1)\left[\sum_{i=1}^{p}\left[\frac{R_{i..}^2}{N_{i..}}\right] - \frac{R_{..}^2}{N}\right]}{(p-1)\left[N - \sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R_{ij.}^2}{N_{ij.}}\right]\right]} \tag{10.2.5}$$

$$\approx F_{(p-1),pq(m-1)}$$

Testing for differences between column effects (j=1,2,....,q)

$$Z_9 = \frac{pq(m-1)\left[\sum_{j=1}^{q}\left[\frac{R_{.j.}^2}{N_{.j.}}\right] - \frac{R_{...}^2}{N}\right]}{(p-1)\left[N - \sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R_{ij.}^2}{N_{ij.}}\right]\right]} \tag{10.2.6}$$

$$\approx F_{(q-1),pq(m-1)}$$

Testing for difference between interaction effects

$$Z_{10} = \frac{pq(m-1)\left[\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R_{ij.}^2}{N_{ij.}}\right] - \sum_{i=1}^{p}\left[\frac{R_{i..}^2}{N_{i.}}\right] - \sum_{j=1}^{q}\left[\frac{R_{.j.}^2}{N_{.j.}}\right] + \frac{R_{...}^2}{N}\right]}{(p-1)(q-1)\left[N - \sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R_{ij.}^2}{N_{ij.}}\right]\right]} \tag{10.2.7}$$

$$\approx F_{(p-1)(q-1),pq(m-1)}$$

10.3 Accuracy of the Associated χ^2 Approximations for the Randomised Complete Block and the Two-way Classification Designs with their Corresponding F Statistics, for Large k

The accuracy of the expressions (10.2.1) and (10.2.4) are determined by simulation methods. Monte Carlo samples from a von Mises distribution with fixed k are made for the distribution specified by the null hypothesis. The computer method used for the generated observations is outlined in Appendix B. As for the testing of the one-way classification and the extended techniques of Watson and Williams in Chapter 6, 10,000 sets of samples of various size were drawn from the von Mises distribution with k = 2,3,4,5 and 10. The same experimental designs investigated in Chapter 6 are used here to enable comparison between tests. For the randomised block three designs are examined varying in size of N.

10.3.1 The Randomised Complete Block Design

Tables 10.3.1 to 10.3.6 examine the chi-squared approximations for each component within the randomised complete block design. Table 10.3.1 gives the first two moments of the components for total and residual measures of variation. The simulated mean value of the chi-squared approximation for each component is seen as a good fit, although slightly over-estimated, and increasing in accuracy as k increases. This follows the findings of Section 9.5. When k is unknown and equation (9.2.5) is used as its replacement the accuracy of the simulated mean value is increased. Again, as in Section 9.4, the simulated variance of the chi-squared approximation of each component is seen to be below its expected value and dependent of the size of N and k. As N and predominantly k increase the accuracy of the variance increases.

Table 10.3.2 shows the accuracy of the total and residual measures of variation components to their associated chi-squared. The goodness of fit in the upper 10 percent significance levels, for concentration parameter greater than 2, is seen to be very good, and increasing in accuracy with increasing size of k. For concentration parameter equal to 2 the approximations under estimate their associated significance levels. Although not shown within the tables, the fit of the simulated distributions in the lower percentile levels is not as good as in the upper tails, but this would be expected given the simulated means and variances of Table 10.3.1.

Tables 10.3.3 and 10.3.4 give the first two moments for the treatment and block measures of variation components, respectively. As for the one-way analysis the simulated mean value of the component is approximately equal to the mean of the chi-squared approximation and increasing in accuracy as k increases. Similarly the variance is seen to be rather poor, most noticeably for small k. As in Section 9.5 the improvement factor β is used and multiplies k in order to increase the accuracy of the chi-squared approximation variance and therefore the distribution fit. The comparable moments are given in adjacent columns in Tables 10.3.3 and 10.3.4 and show a slight increase in the simulated mean value but an important improvement in the accuracy of the variance. This is reflected in the accuracy of the improved treatment and block components to the upper percentage points of their associated chi-squared distributions given in Tables 10.3.5 and 10.3.6.

TABLE 10.3.1 Comparing the mean and variance of the χ^2 approximations for the total and residual measures of variation against their expected values

SAMPLE SIZE	k	$k(N-R^2_{..}/N)$ Mean	(Variance)	SAMPLE SIZE	$k(N-\Sigma(R^2_{i.}/N_{i.})-\Sigma(R^2_{.j}/N_{.j})+R^2_{..}/N)$ Mean	(Variance)
3 by 5	2	14.291	(17.624)	3 by 5	8.208	(9.636)
	3	14.212	(23.696)		8.16	(12.084)
Expected	4	14.174	(26.527)	Expected	8.113	(13.548)
Mean = 14	5	14.109	(25.591	Mean = 8	8.087	(14.006)
Variance = 28	10	14.034	(26.108)	Variance = 16	8.018	(14.411)
3 by 8	2	23.453	(28.414)	3 by 8	14.388	(17.03)
	3	23.51	(39.128)		14.412	(22.319)
Expected	4	23.219	(42.581)	Expected	14.135	(23.987)
Mean = 23	5	23.225	(43.126)	Mean = 14	14.197	(24.369)
Variance = 46	10	23.094	(42.978)	Variance = 28	14.056	(26,003)
5 by 10	2	50.4	(66.342)	5 by 10	37.04	(47.027)
	3	50.331	(87.882)		36.918	(62.739)
Expected	4	49.83	(90.444)	Expected	36.581	(62.181)
Mean = 49	5	49.306	(92.854)	Mean = 36	36.158	(65.149)
Variance = 98	10	49.084	(97.806)	Variance = 72	36.077	(71.385)

TABLE 10.3.2 Examining the accuracy of the χ^2 approximations for the total and residual measures of variation

SAMPLE SIZES	k	$k(N-R^2_{..}/N)$ α 0.90	0.95	0.99	$k(N-\Sigma(R^2_{i.}/N_{i.})-\Sigma(R^2_{.j}/N_{.j})+R^2_{..}/N)$ α 0.90	0.95	0.99
3 by 5	2	0.9296	0.9786	0.9987	0.9312	0.9762	0.9989
	3	.9008	.96	.9972	.9126	.9614	.9968
I=3 treatments	4	.8964	.95	.9914	.9108	.9604	.9946
J=5 blocks	5	.9074	.9562	.993	.9046	.958	.9946
N=15	10	.9008	.9536	.9918	.9044	.9542	.9896
3 by 8	2	0.9268	0.974	0.9985	0.9292	0.973	0.9985
	3	.9026	.9552	.9942	.9098	.9594	.9945
I=3 treatments	4	.90	.9524	.9936	.9092	.9602	.994
J=8 blocks	5	.9054	.9558	.991	.9054	.9588	.9942
N=24	10	.9034	.9532	.9928	.9052	.9512	.9908
5 by 10	2	0.921	0.9712	0.998	0.927	0.9674	0.998
	3	.8876	.9478	.991	.8997	.9508	.9908
I=5 treatments	4	.897	.9494	.9934	.9068	.955	.9936
J=10 blocks	5	.9022	.953	.9918	.9032	.9568	.9929
N=50	10	.9024	.9528	.9916	.9002	.951	.9905

TABLE 10.3.3 Comparing the mean and variance of the χ^2 approximation for the treatment measure of variation against its expected values

SAMPLE SIZE	k	$k(\Sigma(R^2_{.j}/N_{.j})-R^2_{..}/N)$ Mean	(Variance)	$kB(\Sigma(R^2_{.j}/N_{.j})-R^2_{..}/N)$ Mean	(Variance)
3 by 5	2	2.053	(2.432)	2.297	(3.177)
	3	2.061	(3.02)	2.251	(3.551)
Expected	4	2.045	(3.311)	2.155	(3.716)
Mean = 2	5	2.034	(3.416)	2.094	(3.757)
Variance = 4	10	2.028	(3.708)	2.054	(3.889)
3 by 8	2	2.043	(2.374)	2.291	(3.1)
	3	2.055	(2.907)	2.23	(3.418)
Expected	4	1.998	(2.991)	2.106	(3.419)
Mean = 2	5	2.03	(3.309)	2.114	(3.721)
Variance = 4	10	2.012	(3.543)	2.042	(3.781)
5 by 10	2	4.052	(4.616)	4.454	(6.427)
	3	4.092	(5.898)	4.284	(6.934)
Expected	4	4.04	(6.436)	4.217	(7.225)
Mean = 4	5	4.078	(6.857)	4.198	(7.502)
Variance = 8	10	3.996	(7.007)	4.062	(7.551)

TABLE 10.3.4 Comparing the mean and variance of the χ^2 approximation for the block measure of variation against its expected values

SAMPLE SIZE	k	$k(\Sigma(R_{i.}^2/N_{i.})-R_{..}^2/N)$ Mean	(Variance)	$kB(\Sigma(R_{i.}^2/N_{i.})-R_{..}^2/N)$ Mean	(Variance)
3 by 5	2	4.053	(4.69)	4.495	(6.123)
	3	4.068	(6.067)	4.401	(7.133)
Expected	4	4.094	(6.766)	4.31	(7.595)
Mean = 4	5	4.04	(6.941)	4.206	(7.593)
Variance = 8	10	3.999	(7.053)	4.065	(7.784)
3 by 8	2	7.121	(8.348)	8.008	(10.906)
	3	7.206	(10.541)	7.648	(12.396)
Expected	4	7.118	(11.64)	7.308	(13.07)
Mean = 7	5	7.083	(12.342)	7.281	(13.505)
Variance = 14	10	7.028	(12.478)	7.158	(13.518)
5 by 10	2	9.231	(11.432)	10.21	(14.934)
	3	9.174	(12.895)	9.847	(15.162)
Expected	4	9.131	(14.383)	9.514	(16.15)
Mean = 9	5	9.068	(15.743)	9.216	(17.226)
Variance = 18	10	9.032	(15.881)	9.187	(17.369)

TABLE 10.3.5 Examining the accuracy of the χ^2 approximations for the treatment measure of variation

SAMPLE SIZE	k	$K(\Sigma(R^2_{.j}/N_{.j})-R^2_{..}/N)$ α 0.90	0.95	0.99	$kB(\Sigma(R^2_{.j}/N_{.j})-R^2_{..}/N)$ α 0.90	0.95	0.99
3 by 5	2	0.9246	0.9746	0.9984	0.8878	0.9518	0.9944
	3	.914	.9626	.9942	.8888	.9495	.9908
I=3 treatments	4	.9082	.9618	.9922	.8936	.953	.9904
J=5 blocks	5	.9062	.9578	.9918	.8964	.9518	.9902
N=15	10	.9008	.9518	.9914	.8994	.9488	.9896
3 by 8	2	0.933	0.974	0.9976	0.9034	0.9556	0.9936
	3	.9154	.966	.9956	.8922	.9516	.992
I=3 treatments	4	.9192	.9644	.9946	.9068	.9552	.9934
J=8 blocks	5	.9106	.956	.9932	.901	.9498	.9914
N=24	10	.9084	.9552	.9922	.9038	.9534	.9912
5 by 10	2	0.9318	0.975	0.9976	0.8914	0.9534	0.9928
	3	.9163	.9638	.994	.8916	.9478	.9908
I=5 treatments	4	.916	.959	.9934	.8978	.9496	.9906
J=10 blocks	5	.9082	.9552	.9922	.8918	.9468	.9906
N=50	10	.908	.9596	.9928	.903	.9554	.9916

TABLE 10.3.6 Examining the accuracy of the χ^2 approximations for the block measure of variation

SAMPLE SIZE	k	$k(\Sigma(R_{i.}^2/N_{i.})-R_{..}^2/N$			$kB(\Sigma(R_{i.}^2/N_{i.})^2-R_{..}^2/N$		
		α = 0.90	0.95	0.99	0.90	0.95	0.99
3 by 5	2	0.9348	0.9796	0.9982	0.89	0.9534	0.9934
	3	.9164	.9682	.9952	.8904	.9506	.991
I=3 treatments	4	.905	.9574	.993	.888	.9477	.9904
J=5 blocks	5	.9082	.9562	.9924	.8971	.9502	.9903
N=15	10	.9074	.9566	.9924	.8998	.9509	.9912
3 by 8	2	0.9341	0.9797	0.9984	0.8827	0.9394	0.9928
	3	.9153	,9614	.9965	.8872	.948	.9924
I=3 treatments	4	.9072	.9606	.9973	.888	.948	.9904
J=8 blocks	5	.9106	.9562	.992	.899	.9464	.9898
N=24	10	.9056	.9564	.9942	.8992	.951	.9917
5 by 10	2	0.93	0.9714	0.997	0.8694	0.9394	0.9898
	3	.9258	.9696	.9954	.883	.9508	.9912
I=5 treatments	4	.911	.9602	.995	.8894	.9461	.9906
J=10 blocks	5	.9095	.9532	.9916	.8911	.9461	.9892
N=50	10	.907	.9541	.9912	.90	.951	.9907

The accuracy of the F distribution approximations are provided in Tables 10.3.7 and 10.3.8. Table 10.3.7 examines the F distribution statistic for the testing of the p treatments Z_6, against its improved F distribution where the term β has been included. Table 10.3.8 examines the corresponding test for q blocks, Z_7. The improvement tests show better fits for both factors across all significance levels, although it is important to note that it still slightly under–estimates the proportion α leading to an increased probability of a type II error.

In Chapters 6 and 7 the extension of Watson and Williams and Stephens approaches to larger experimental designs was shown to breakdown due to the combination of sample means. Examination of the chi–squared approximations for randomised complete block and two–way classification designs were, however, carried out since the combination problem was negligible for very large k and did not affect the associated test statistics. On the assumption that the extended techniques were valid, direct comparison may be made to the corresponding tables in this section. The associated tables for the randomised complete block design are Tables 6.4.1 to 6.4.7. For all the components within (10.2.1), incorporating the correction factor β, show a comparable fit for large k and an improved fit for small k for both the chi–squared approximation and moment calculations.

TABLE 10.3.7 Comparing the accuracy of the test statistic Z_6 against its improved statistic for examining treatment effects

SAMPLE SIZE	k	Z_6 Equation (10.2.2) α			BZ_6 α		
		0.90	0.95	0.99	0.90	0.95	0.99
3 by 5	2	0.9306	0.9685	0.9964	0.9135	0.9625	0.9955
	3	.9212	.9683	.9955	.91	.9618	.9931
	4	.9141	.9601	.9939	.906	.9584	.9935
	5	.9107	.9571	.991	.9051	.9532	.9908
	10	.9036	.9557	.9925	.90	.9524	.992
3 by 8	2	0.9313	0.9683	0.9959	0.9106	0.9611	0.9945
	3	.9214	.9648	.9951	.908	.9586	.9924
	4	.919	.9591	.9936	.9061	.9552	.994
	5	.9089	.9561	.9918	.9042	.9521	.9915
	10	.904	.9538	.9906	.9022	.9519	.99
5 by 10	2	0.9298	0.9671	0.9959	0.9141	0.9601	0.9919
	3	.9211	.9651	.9943	.9012	.9577	.9922
	4	.92	.9597	.9931	.9052	.9547	.9925
	5	.9106	.9547	.992	.8997	.9494	.9907
	10	.9034	.9539	.9927	.9023	.9518	.9923

TABLE 10.3.8 Comparing the accuracy of the test statistic Z_7 against its improved statistic for examining block effects

SAMPLE SIZE	k	Z_7 Equation (10.2.3) α			BZ_7 α		
		0.90	0.95	0.99	0.90	0.95	0.99
3 by 5	2	0.9438	0.9748	0.9976	0.9238	0.971	0.996
	3	.9318	.9671	.9941	.9208	.9604	.993
	4	.9201	.961	.9937	.9092	.9541	.9932
	5	.9128	.9585	.9937	.9064	.9543	.9914
	10	.9118	.9579	.9914	.905	.954	.9912
3 by 8	2	0.9398	0.9698	0.9964	0.9103	0.9634	0.993
	3	.9271	.9667	.9937	.9059	.959	.9925
	4	.9231	.9612	.9929	.9101	.9552	.9919
	5	.9103	.959	.993	.9042	.9531	.992
	10	.905	.955	.991	.901	.9522	.991
5 by 10	2	0.9367	0.9682	0.9961	0.9106	0.953	0.9919
	3	.929	.9669	.9949	.9101	.9608	.9931
	4	.9214	.9614	.9942	.9018	.954	.9926
	5	.9107	.9587	.992	.8991	.9514	.9894
	10	.909	.9562	.9926	.907	.9527	.9914

10.3.2 Two-way Classification with Interaction

As for the randomised complete block comparable statistics for the two-way classification with interaction design, examining the component chi-squared approximations, are produced in the six tables from 10.3.9 to 10.3.14. Here the chi-squared approximation moments are not reproduced as similar terms from other models have already indicated their adequacy. Similarly the correction factor β has been included for the main effect and interaction terms.

There are four two-way designs simulated, each varying in size from N=30 to N=90 and varying in concentration parameter from k=2 to k=10. Table 10.3.9 indicates the accuracy of the total measure of variation and the first main effect to their respective chi-squareds for 10,000 simulations. Table 10.3.10 for the second main effect and interaction terms and finally Table 10.3.11 for the residual term. The main effect and interaction terms show excellent fits across all values of k, while the total and residual terms are relatively poor fits for concentration parameter as low as 2.

The accuracy of the three test statistics Z_8, Z_9 and Z_{10} are shown in Tables 10.3.12, 10.3.13 and 10.3.14 respectively, with the corresponding improvement factor β applied. The F distribution approximations for the main effects show similar good fits as in the one-way classification and randomised block designs, with the accuracy decreasing at $k\approx 2$. The final Table, 10.3.14, indicates the accuracy of the interaction test statistic to the F distribution and shows an excellent fit across all concentration parameters.

Comparing the Tables 10.3.9 to 10.3.14 with the corresponding Tables 6.4.8 to 6.4.13 of Chapter 6, and on the assumption that the extended techniques are valid for large k, shows once again an improvement in the chi-squared accuracy for the new approach.

TABLE 10.3.9 Examining the accuracy of the total and first main effect components to their respective χ^2 distributions

SAMPLE SIZE	k	$k(N-R^2_{...}/N)$ α = 0.90	0.95	0.99	$kB(\Sigma(R^2_{i..}/N_{i..})-R^2_{...}/N)$ α = 0.90	0.95	0.99
2 by 3 (by 5)	2	0.9296	0.9711	0.9984	0.8964	0.9598	0.9948
	3	.8944	.9612	.9954	.9014	.9558	.9926
(2 by 3 design	4	.90	.9538	.9944	.90	.9546	.9936
with 5 observa-	5	.9042	.957	.9924	.8974	.9498	.9912
tions within	10	.905	.956	.9898	.9054	.9544	.9914
each cell)N=30							
2 by 3(by 10)	2	0.9191	0.9695	0.997	0.8984	0.954	0.996
	3	.883	.9472	.991	.8946	.9526	.9938
N=60	4	.8946	.9494	.9903	.8954	.9498	.9912
	5	.899	.9582	.991	.9007	.9551	.9912
	10	.9071	.9535	.9896	.9033	.9548	.9911
3 by 3 (by 5)	2	0.9253	0.969	0.9981	0.90	0.961	0.9937
	3	.8982	.9532	.9928	.9057	.9576	.9943
N=45	4	.897	.9522	.9934	.9052	.957	.9913
	5	.8954	.9502	.993	.9008	.9518	.9918
	10	.899	.954	.9906	.9059	.9546	.9925
3 by 3 (by 10)	2	0.9106	0.966	0.9967	0.8908	0.9523	0.9928
	3	.8892	.9484	.99	.9028	.9561	.9932
N=90	4	.8912	.9466	.9934	.9043	.955	.9931
	5	.9002	.9491	.9917	.8999	.9506	.99
	10	.9031	.9528	.9928	.9034	.9505	.9909

TABLE 10.3.10 Examining the accuracy of the second main effect and interaction components to their respective χ^2 distributions

SAMPLE SIZE	k	$B(\Sigma(R^2_{.j.}/N_{.j.})-R^2_{...}/N)$ α 0.90	0.95	0.99	$kB(\Sigma\Sigma(R^2_{ij.}/N_{ij.})-\Sigma(R^2_{i..}/N_{i..})-\Sigma(R^2_{.j.}/N_{.j.})+R^2_{..}$ α 0.90	0.95	0.99
2 by 3 (by 5)	2	0.9036	0.9581	0.9957	0.8951	0.9565	0.9946
	3	.8951	.95	.9917	.9011	.9593	.9936
(2 by 3 design	4	.90	.952	.9935	.8963	.9553	.9917
with 5 observa-	5	.9071	.9522	.9923	.8942	.9522	.9931
tions within	10	.9056	.953	.9908	.9005	.9553	.99
each cell)N=30							
2 by 3 (by 10)	2	0.9002	0.9537	0.9958	0.9004	0.9599	0.9957
	3	.9056	.9545	.9914	.8933	.9518	.9924
	4	.9014	.9507	.9904	.8988	.9543	.9921
	5	.902	.9511	.9923	.9023	.9526	.9903
	10	.906	.9565	.9905	.9031	.9527	.9911
3 by 3 (by 5)	2	0.8916	0.9555	0.996	0.8837	0.9519	0.9923
	3	.8953	.9556	.9937	.8889	.9502	.9908
	4	.8992	.9496	.9923	.8911	.9472	.9901
	5	.9015	.9524	.9921	.902	.9529	.9904
	10	.906	.9491	.9915	.9018	.9508	.9896
3 by 3 (by 10)	2	0.9001	0.9559	0.9928	0.8951	0.9487	0.9936
	3	.895	.9521	.9917	.8899	.9451	.9914
	4	.9055	.9541	.9924	.8994	.9512	.9933
	5	.8978	.9527	.9909	.9001	.9521	.9923
	10	.90	.9513	.9896	.9007	,9517	.99

TABLE 10.3.11 Examining the accuracy of the residual component to its χ^2 distribution with pq(m-1) degrees of freedom

SAMPLE SIZE	k	$k(N-\Sigma\Sigma(R^2_{ij.}/N_{ij.}))$ α 0.90	0.95	0.99
2 by 3 (by 5)	2	0.9303	0.976	0.9985
	3	.9036	.958	.9954
(2 by 3 design	4	.9052	.9586	.9948
with 5 observa-	5	.9046	.9561	.9929
tions within	10	.9045	.9541	.9902
each cell)N=30				
2 by 3 (by 10)	2	0.9214	0.9691	0.9969
	3	.8898	.9495	.9921
N=60	4	.8975	.9512	.9906
	5	.9035	.9553	.9931
	10	.9012	.9531	.9889
3 by 3 (by 5)	2	0.9284	0.9741	0.9974
	3	.9026	.9602	.9951
N=45	4	.9066	.9549	.9935
	5	.9016	.9541	.9923
	10	.8965	.9515	.9917
3 by 3 (by 10)	2	0.9103	0.9647	0.9954
	3	.8871	.948	.9912
N=90	4	.8878	.9463	.9917
	5	.8982	.951	.9919
	10	.9043	.9549	.9913

TABLE 10.3.12 Comparing the accuracy of the test statistic Z_8 against its improved statistic for examining the first main effect, p

SAMPLE SIZE	k	Z_8 Equation (10.2.5) α			BZ_8 α		
		0.90	0.95	0.99	0.90	0.95	0.99
2 by 3 (by 5)	2	0.9357	0.975	0.9973	0.9103	0.9606	0.9941
	3	.9138	.9681	.9961	.8999	.9581	.9929
N=30	4	.9157	.9635	.9935	.904	.9552	.9919
	5	.9164	.9591	.9936	.9084	.955	.9925
	10	.9062	.9532	.9917	.903	.9518	.9907
2 by 3 (by 10)	2	0.9318	0.9729	0.9975	0.9098	0.9589	0.9955
	3	.9245	.9662	.9945	.9101	.9552	.9912
N=60	4	.9097	.958	.9926	.9013	.9562	.9906
	5	.9096	.9621	.994	.9009	.9556	.992
	10	.9099	.955	.992	.906	.953	.9908
3 by 3 (by 5)	2	0.9342	0.9738	0.9982	0.9063	0.9601	0.9957
	3	.9242	.971	.9952	.9044	.9592	.9925
N=45	4	.9174	.9601	.993	.9032	.9571	.9907
	5	.9143	.9608	.9918	.9046	.9541	.9932
	10	.9079	.9537	.9926	.8981	.9496	.9902
3 by 3 (by 10)	2	0.9391	0.9721	0.9969	0.9107	0.9608	0.9935
	3	.9201	.9662	.9942	.9051	.9521	.9921
N=90	4	.9087	.9574	.9923	.9072	.953	.9914
	5	.9092	.9589	.9938	.90	.9541	.9917
	10	.9042	.9521	.9908	.8991	.9486	.99

TABLE 10.3.13 Comparing the accuracy of the test statistic Z_g against its improved statistic for examining the second main effect, q

SAMPLE SIZES	k	Z_g Equation (10.2.6) α 0.90	0.95	0.99	BZ_g α 0.90	0.95	0.99
2 by 3 (by 5)	2	0.9395	0.9796	0.9978	0.9118	0.9628	0.9942
	3	.9285	.9704	.997	.908	.96	.9938
N=30	4	.919	.9622	.9928	.9058	.9548	.9909
	5	.9107	.9561	.9926	.9042	.9508	.9907
	10	.908	.9561	.9931	.9029	.9542	.991
2 by 3 (by 10)	2	0.9378	0.9775	0.9987	0.9191	0.9586	0.9956
	3	.9251	.9705	.996	.9047	.9603	.9935
N=60	4	.9168	.9655	.9933	.9049	.9551	.9911
	5	.9168	.96	.9931	.9074	.9552	.9922
	10	.9087	.9601	.9921	.9031	.9546	.9913
3 by 3 (by 5)	2	0.9385	0.9781	0.9982	0.9137	0.961	0.9937
	3	.9262	.9694	.9964	.9102	.9601	.9941
N=45	4	.9175	.9642	.9941	.91	.9568	.9927
	5	.9111	.9597	.992	.9051	.9548	.9919
	10	.9071	.9568	.9933	.9051	.9551	.9921
3 by 3 (by 10)	2	0.9374	0.9761	0.9972	0.9044	0.9573	0.9931
	3	.9271	.9642	.9963	.9131	.9632	.993
N=90	4	.9168	.9658	.9932	.9101	.9563	.9932
	5	.9098	.9569	.9931	.9025	.9524	.9909
	10	.9067	.9561	.9919	.904	.9514	.9921

TABLE 10.3.14 Comparing the accuracy of the test statistic Z_{10} against the improved statistic for examining the interaction effect

SAMPLE SIZES	k	Z_{10} Equation (10.2.7)			BZ_{10}		
		α			α		
		0.90	0.95	0.99	0.90	0.95	0.99
2 by 3 (by 5)	2	0.9402	0.9771	0.9976	0.9097	0.9605	0.9937
	3	.9331	.9712	.9959	.9114	.9611	.9937
N=30	4	.9156	.9577	.9931	.9026	.9519	.9918
	5	.9131	.9574	.9951	.9047	.9516	.9936
	10	.9067	.9521	.9901	.9031	.9494	.9894
2 by 3 (by 19)	2	0.9331	0.9805	0.9981	0.9115	0.9598	0.9951
	3	.9236	.9681	.9961	.9046	.9565	.9941
N=60	4	.9143	.9627	.9947	.9029	.9524	.9912
	5	.9131	.9602	.993	.9091	.9543	.9911
	10	.9101	.9548	.9922	.9057	.9523	.9907
3 by 3 (by 5)	2	0.9382	0.982	0.997	0.8997	0.9571	0.9939
	3	.9284	.9683	.9972	.9002	.9556	.9925
N=45	4	.9157	.9597	.9945	.8987	.9504	.9897
	5	.9138	.9612	.9945	.8995	.9548	.9927
	10	.9083	.9543	.9927	.8989	.9521	.9914
3 by 3 (by 10)	2	0.937	0.9767	0.9951	0.9005	0.9567	0.9938
	3	.9222	.9661	.9969	.8989	.9537	.9921
N=90	4	.9142	.9602	.994	.9035	.9576	.9935
	5	.9149	.9591	.9932	.9051	.955	.9905
	10	.9051	.953	.9919	.9997	.953	.9906

10.4 Randomised Complete Block and Two-way Classification with Interaction Designs for Small k

As discussed in Sections 9.7 and 9.8 only the associated component chi-squared approximation may be used as a test of difference, since the corresponding total and residual components, for small concentration parameter, do not form chi-squared distributions. For the randomised complete block and two-way analysis designs with small k the models may be seen as given in (10.2.1) and (10.2.4) but the total and residual terms may not be assumed to be chi-squared distributed. From (10.2.1), (10.2.4) and Section 9.2 the test statistic to examine the null hypothesis that there is no difference between the p treatments gives:

$$Z_{11} = \frac{2}{1 - \rho^2}\left\{\sum_{i=1}^{p}\left[\frac{R_{i..}^2}{N_{i..}}\right] - \frac{R_{...}^2}{N}\right\} \tag{10.4.1}$$

which has a chi-squared distribution with 2(p−1) degrees of freedom.

Similarly, the test statistic Z_{12} will be produced to test the null hypothesis that there is no difference between the q blocks within the randomised complete block, or q treatments within the two-way design:

$$Z_{12} = \frac{2}{1 - \rho^2}\left\{\sum_{j=1}^{q}\left[\frac{R_{.j.}^2}{N_{.j.}}\right] - \frac{R_{...}^2}{N}\right\} \tag{10.4.2}$$

which has a chi-squared distribution with 2(q−1) degrees of freedom. Testing for differences between interaction terms within the two-way analysis gives test statistic (10.4.3)

$$Z_{13} = \frac{2}{1 - \rho^2}\left\{\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R_{ij.}^2}{N_{ij.}}\right] - \sum_{i=1}^{p}\left[\frac{R_{i..}^2}{N_{i..}}\right] - \sum_{j=1}^{q}\left[\frac{R_{.j.}^2}{N_{.j.}}\right] + \frac{R_{...}^2}{N}\right\} \tag{10.4.3}$$

$$\approx \chi^2_{2(p-1)(q-1)}$$

As for large concentration parameter, test statistics (10.4.1), (10.4.2) and (10.4.3) indicate whether or not the mean directions are equal, they do not allow for discrimination amongst single mean directions.

10.5 Accuracy of the Associated Chi-Squared Approximations for Small k

The accuracy of the test statistics (10.4.1), (10.4.2) and (10.4.3) are determined by simulation methods. The Monte Carlo samples from a von Mises distribution are generated by the computer method described in Appendix B. As for the testing of large k, 10,000 sets of samples of various size were drawn with varying concentration parameter values. Before the tests are applied the equality of concentration parameter is checked using test statistic (4.5.2). Ranges of small k are given as its accuracy cannot be assumed for such small values. The test statistics are examined with equivalent sample sizes and components within the two-way classification, since the main effects for both the randomised block and two-way designs are similar.

Table 10.5.1 gives the first two moments of the chi-squared approximations for the two main effects within the two-way design. The simulated mean value for each component is seen to be a very good fit to the mean of the chi-squared approximation. The variances are also good fits, increasing in accuracy as the size of sample and k increase. Table 10.5.2 shows the accuracy of the first two moments for the interaction term to the expected values. As for the main effects the simulated means and variances are comparable to their associated chi-squared approximations notably increasing in accuracy as N and predominantly k increase.

Tables 10.5.3 and 10.5.4 compare the accuracy of the test statistics, Z_{11}, Z_{12} and Z_{13} to their corresponding chi-squared distributions in the upper 10 percent significant levels. The goodness of fit for the tests with concentration parameter less than 1 are seen to be very good for all the test statistics, and as for all previous goodness of fit examinations, the accuracy is seen to increase as the size of sample increases.

Tables 10.5.5 and 10.5.6 take the test statistics out of their proven range and examine their accuracy for concentration parameter values of 1.25 and 1.75. The test statistic for both main effects and interaction terms are tested. The tables clearly indicate that the test statistics show very good fits to the upper 10 percent significance levels and are reliable tests for experimental designs with concentration parameters in the range 1 to 2.

TABLE 10.5.1 Comparing the mean and variance of the χ^2 approximations with (p-1) and (q-1) degrees of freedom to their expected values, for small k

SAMPLE SIZE	k	$2/(1-\rho^2)(\Sigma(R^2_{i..}/N_{i..})-(R^2_{...}/N))$ Mean	(Variance)	SAMPLE SIZE	$2/(1-\rho^2)(\Sigma(R^2_{.j.}/N_{.j.})-(R^2_{...}/N))$ Mean	(Variance)
2 by 3 (by 5)				2 by 3 (by 5)		
Expected	0.0-0.5	4.006	(7.584)	Expected	1.984	(3.899)
Mean = 4	0.5-1.0	3.967	(7.558)	Mean = 2	1.975	(4.175)
Variance = 8				Variance = 4		
2 by 3 (by 10)						
Expected	0.0-0.5	4.034	(7.774)	Expected	2.009	(4.241)
Mean = 4	0.5-1.0	4.082	(8.376)	Mean = 2	1.969	(3.752)
Variance = 8				Variance = 4		
3 by 3 (by 5)						
Expected	0.0-0.5	3.95	(7.529)	Expected	4.017	(7.545)
Mean = 4	0.5-1.0	4.005	(7.8)	Mean = 4	4.046	(8.089)
Variance = 8				Variance = 8		
3 by 3 (by 10)						
Expected	0.0-0.5	4.006	(8.019)	Expected	4.010	(8.204)
Mean = 4	0.5-1.0	3.973	(7.831)	Mean = 4	4.021	(7.661)
Variance = 8				Variance = 8		

TABLE 10.5.2 Comparing the mean and variance of the χ^2 approximation with $(p-1)(q-1)$ degrees of freedom to their expected values

SAMPLE SIZE	k	$2/(1-\rho^2)(\Sigma\Sigma(R^2_{ij.}/N_{ij.}) - \Sigma(R^2_{i..}/N_{i..}) - \Sigma(R^2_{.j.}/N_{.j.}) + (R^2_{...}/N))$ Mean	(Variance)
2 by 3 (by 5)			
Expected	0.0-0.5	3.967	(7.422)
Mean = 4	0.5-1.0	3.957	(7.839)
Variance = 8			
2 by 3 (by 10)			
Expected	0.0-0.5	3.992	(7.589)
Mean = 4	0.5-1.0	4.001	(8.092)
Variance = 8			
3 by 3 (by 5)			
Expected	0.0-0.5	7.888	14.043
Mean = 8	0.5-1.0	8.045	(15.525)
Variance = 16			
3 by 3 (by 10)			
Expected	0.0-0.5	7.971	(15.840)
Mean = 8	0.5-1.0	8.179	(16.312)
Variance = 16			

TABLE 10.5.3 Examining the accuracy of the χ^2 approximations for the two main effect components, for small k

SAMPLE SIZE	k	$2/(1-\rho^2)(\Sigma(R^2_{i..}/N_{i..})-R^2_{...}/N)$			$2/(1-\rho^2)(\Sigma(R^2_{.j.}/N_{.j.})-R^2_{...}/N)$		
		α			α		
		0.90	0.95	0.99	0.90	0.95	0.99
2 by 3 (by 5) (2 by 3 design with 5 observations within each cell) N=30	0.0-0.5	0.9018	0.9518	0.993	0.8986	0.9530	0.9904
	0.5-1.0	.9092	.9561	0.9913	.9023	.9458	.989
2 by 3 (by 10) N=60	0.0-0.5	0.905	0.9508	0.9906	0.899	0.9496	0.9868
	0.5-1.0	.8957	.9491	.9877	.9116	.9598	.9907
3 by 3 (by 5) N=45	0.0-0.5	0.9038	0.954	0.9916	0.9006	0.9522	0.9918
	0.5-1.0	.90	.9497	.9912	.899	.9501	.99
3 by 3 (by 10) N=90	0.0-0.5	0.9002	0.9462	0.989	0.8984	0.951	0.9906
	0.5-1.0	.9017	.9503	.9897	.9063	.9551	.9907

TABLE 10.5.4 Examining the accuracy of the χ^2 approximation for the interaction component for small k

SAMPLE SIZE	k	$2/(1-\rho^2)(\Sigma\Sigma(R^2_{ij.}/N_{ij.})-\Sigma(R^2_{i..}/N_{i..})-\Sigma(R^2_{.j.}/N_{.j.})+(R^2_{...}/N)$		
		α		
		0.90	0.95	0.99
2 by 3 (by 5) (2 by 3 design with 5 observations within each cell)N=30	0.0-0.5	0.907	0.955	0.9925
	0.5-1.0	.9053	.9526	.9897
2 by 3 (by 19) N=60	0.0-0.5	0.9006	0.9561	0.9928
	0.5-1.0	.8964	.9497	.9897
3 by 3 (by 5) N=45	0.0-0.5	0.91	0.9599	0.9932
	0.5-1.0	.894	.9473	.9893
3 by 3 (by 10) N=90	0.0-0.5	0.9072	0.9544	0.9923
	0.5-1.0	.8924	.9516	.992

TABLE 10.5.5 Examining the accuracy of the χ^2 approximations for the two main effect components, for k in the range 1 to 2

SAMPLE SIZE	k	$2/(1-\rho^2)((R^2_{i..}/N_{i..})-R^2_{...}/N)$ α 0.90	0.95	0.99	$2/(1-\rho^2)((R^2_{.j.}/N_{.j.})-R^2_{...}/N)$ α 0.90	0.95	0.99
2 by 3 (by 5) (2 by 3 design with 5 observations within each cell) N=30	1.25	0.8841	0.9451	0.9916	0.8948	0.9454	0.9881
	1.75	.8892	.943	.9874	.8951	.9463	.9897
2 by 3 (by 10) N=60	1.25	0.8878	0.948	0.9887	0.9007	0.9497	0.989
	1.75	.8859	.9463	.9803	.8952	.9473	.9811
3 by 3 (by 5) N=45	1.25	0.8891	0.9474	0.987	0.8891	0.9385	0.9851
	1.75	.8909	.9441	.9852	.8904	.9414	.9853
3 by 3 (by 10) N=90	1.25	0.8977	0.9501	0.9913	0.899	0.9486	0.9872
	1.75	.8903	.945	.9853	.9014	.9509	.9874

TABLE 10.5.6 Examining the accuracy of the χ^2 approximations for the interaction component, for k in the range 1 to 2

SAMPLE SIZE	k	$2/(1-\rho^2)(\Sigma\Sigma(R^2_{ij.}/N_{ij.})-\Sigma(R^2_{i..}/N_{i..})-\Sigma(R^2_{.j.}/N_{.j.})+(R^2_{...}/N))$		
		α 0.90	0.95	0.99
2 by 3 (by 5) (2 by 3 design with 5 observations within each cell) N=30	1.25	0.9017	0.9503	0.9885
	1.75	.9024	.9491	.9879
2 by 3 (by 10) N=60	1.25	0.8951	0.9396	0.9792
	1.75	.8973	.9443	.985
3 by 3 (by 5) N=45	1.25	0.8967	0.9471	0.9866
	1.75	.897	.9463	.9842
3 by 3 (by 10) N=90	1.25	0.902	0.9501	0.9895
	1.75	.9013	.9497	.9882

10.6 Summary

This Chapter has investigated the adequacy of the generalised approach for the randomised complete block and two-way classification designs, with particular reference to their accuracy compared to the associated chi-squared and F approximations. The randomised complete block and two-way classification designs have been used to illustrate the validity of the approach for larger more complex design situations.

Section 10.3 has show that for large k ($k \geqslant 2$), and with the inclusion of the correction factor β, the new test statistics are reliable and show excellent approximations to their associated F distributions. For small k only the main factors may be tested against their corresponding chi-squared distributions, nevertheless, the tests show excellent fits for a concentration parameter range of 0 to 1.

Of most interest is the examination of the test statistics in the concentration parameter range 1 to 2. Ideally the test statistics for large k would be desirable since measures of total and residual variation may be found and utilised. However, the test statistics for small k show very good distribution fits and indicate that these tests should be used for all experimental situations where the concentration parameter is in the range 0 to 2.

CHAPTER 11

ANALYSIS OF VARIANCE EXAMPLES FOR CIRCULAR STATISTICS

11.1 Introduction

Having discounted the possibility of extending the previous one-way analysis of variance approach to larger designs, we will now proceed to test the new approaches and their validity with real data sets. The designs range from the simple one-way design to the Graeco-Latin square and split plot designs. This section not only demonstrates the application of the new procedures but also indicates how they may be used to analyses other designs not discussed in this thesis.

11.2 One-way Analysis

Example 11.2.1 for Large k

For a one-way classification analysis with large k an example given by Gadsden and Kanji (1983) will be used. The examination concerns the orientation of particles in clay strata observed from photographs for various magnifications. The data, given in Table 11.2.1, has been reproduced from Gadsden and Kanji (1983).

Table 11.2.1 Raw Data From Photographs

Magnification:	
100	82,71,85,89,78,77,74,71.68,83,72,73,81,65,62,90, 92,80,77,93,75,80,69,74,77,75,71,82,84,79,78,81, 89,79,82,81,85,76,71,80,94,68,72,70,59,80,86,98, 82,73
200	75,74,71,63,83,74,82,78,87,87,82,71,60,66,63,85, 81,78,80,89,82,82,92,80,81,74,90,78,73,72,80,59, 64,78,73,70,79,79,77,81,72,76,69,73,75,84,81,51, 76,88
400	70,76,79,86,77,86,77,90,88,82,84,70,87,61,71,89, 72,90,74,88,82,68,83,75,90,79,89,78,74,73,71,80, 83,89,68,81,47,88,69,76,71,67,76,90,84,70,80,77, 93,89
1200	73,90,72,91,73,79,82,87,78,83,74,82,85,75,67,72, 78,88,89,71,73,77,90,82,80,81,89,87,78,73,78,86, 73,84,68,75,70,89,54,80,90,88,81,82,88,82,75,79, 83,82
400x1.3	88,69,64,78,71,68,54,80,73,72,65,73,93,84,80,49, 78,82,95,69,87,83,52,79,85,67,82,84,87,83,88,79, 83,77,78,89,75,72,88,78,62,68,89,74,71,73,84,56, 77,71

For the photograph magnifications we wish to test the null hypothesis

$$H_0 : \mu_{0,1} = \mu_{0,2} = \ldots\ldots = \mu_{0,q}$$

against the alternative hypothesis that at least one of the equalities does not hold. Prior to testing the hypothesis it is necessary to test the assumptions that (a) the samples are drawn from a von Mises population and (b) the concentration parameter k has the same value in each sample. Each of the sample populations have been tested by Watsons U^2 statistic (1961), using the critical values supplied by Stephens (1964), to show von Mises distributed data sets. Similarly, the homogeneity of the concentration parameters have been tested and validated via test statistic (4.5.5) (U_3= 5.584 distributed as χ^2_4).

The one-way analysis components of variation are given by:

$$k\left[N - \frac{R^2_{..}}{N}\right] = k\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2_{..}}{N}\right] + k\left[N - \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right]\right] \qquad (11.2.1)$$

and the modified test for the null hypothesis will provide an F-ratio:

$$F_{(q-1)(N-q)} = \beta\left\{\frac{(N-q)\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2_{..}}{N}\right]}{(q-1)\left[(N - \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right]\right]}\right\} \qquad (11.2.2)$$

The analysis of variance for Gadsden and Kanji's data is given in Table 11.2.2.

Table 11.2.2 Analysis of Variance Table

Source of Variation	d.f.	Measure of Variation
Between Photographs	q-1	$\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2_{..}}{N}$
Within Photographs	N-q	$N - \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right]$
Total	N-1	$N - \frac{R^2_{..}}{N}$

Statistics $\quad \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] = 228.1931 \qquad \frac{R^2_{..}}{N} = 227.6244$

$q=5 \qquad N_{.1} = N_{.2} = \ldots\ldots = N_{.q} = 50 \qquad N = 250$

which gives the following ANOVA table:

Table 11.2.3 Analysis of Variance Table:

Source of Variation	d.f.	Measure of Variation	Mean MV	F
Between Photographs	4	0.5687	0.14217	1.5974
Within Photographs	245	21.8069	0.089	
Total	249	22.3756		

$$\hat{k} = 11.02 \qquad \frac{1}{\beta} = 1 - \frac{1}{5\hat{k}} - \frac{1}{10\hat{k}^2} \quad \text{or } \beta = 1.01959$$

The value of $\hat{k}$ is found via approximation (3.3.11) for large k. The modified F' value is $\beta \times F_{4,245} = 1.628$; and as the table value of $F_{4,245}(0.05) = 2.37$, the result is not significant and we can conclude (like Gadsden and Kanji) that there is no observed significant difference between the orientation of the clay particles under differing photographic magnifications.

Example 11.2.2 for Small k

An example of the test procedure for small k is given by using Mardia's (1972) example on wind directions in degrees at Gorleston, England, at 11hr – 12hr on Sundays in 1968 classified according to the four seasons. The data has been reproduced from Mardia (1972).

Table 11.2.4 Wind Directions in Degrees at Gorleston on Sundays in 1968 According to the Four Seasons

Season	Wind Directions in Degrees
Winter	50,120,190,210,220,250,260,290,290,320,320,340
Spring	0,20,40,60,160,170,200,220,270,290,340,350
Summer	10,10,20,20,30,30,40,150,150,150,170,190,290
Autumn	30,70,110,170,180,190,240,250,260,260,290,350

Do the wind directions for the four seasons differ significantly within the given data set? In this case the test statistic to be used is given by;

$$\chi^2_{2(q-1)} = \frac{2}{1-\rho^2}\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2_{..}}{N_{..}}\right] \tag{11.2.3}$$

The associated circular statistics are given in Table 11.2.5

Table 11.2.5 Statistics for the Wind Directions

Season	$N_{.j}$	$R_{.j}$	$\hat{\mu}_{0,j}$	$\hat{k}_{.j}$
Winter	12	5.1185	272°	0.94
Spring	12	2.1321	330°	0.36
Summer	13	3.8680	57°	0.62
Autumn	12	3.1878	232°	0.55
Combined Sample	49	5.8771	292°	0.24

$$\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] = 4.5598 \qquad \frac{R^2_{..}}{N} = 0.7049$$

The values of $\hat{k}_{.j}$ are found via the 'simple' approximation (3.2.11)

$$\hat{k}_0 = 2\left[\frac{R}{N}\right] \qquad \hat{k}_{.j} = 2\left[\frac{R_{.j}}{N_{.j}}\right] \qquad j = 1,2, \ldots\ldots,q$$

Prior to examining any difference between the four seasons the assumption that the concentration parameters are equal must be tested. Test statistic (4.5.2) is used to test their homogeneity and was also used by Mardia (1972). A statistic value of $U_1 = 0.6013$ (distributed as χ^2_3) indicates that the concentration parameters for the wind directions may be regarded as homogeneous.

Applying the test statistic leads to

$$\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2_{..}}{N} = 3.8549$$

$$\rho = \frac{I_1(\hat{k})}{I_0(\hat{k})} \quad \text{and} \quad \frac{2}{1-\rho^2} = 2.0288$$

Hence

$$\frac{2}{1-\rho^2}\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2_{..}}{N}\right] = 7.82$$

Table value χ^2_6 (0.05) = 12.59

Therefore the wind direction for the four seasons are seen not to be significantly different (the same solution as given by Mardia (1972)).

Example 11.3.1 for Large k

This example is taken from the field of clinical psychology and is an example given by Ramano (1977). A psychiatrist was aware that certain organic compounds in the fluid surrounding the brain will, when purified and separately placed into solution rotate the plane of a polarized light source. The psychiatrist was interested in determining whether the optical activity of a specific compound was measurably different for various degrees of schizophrenia. Five distinct levels of schizophrenic behaviour were recognised and for each level 4 patients were selected. Measurements were taken as to the extent to which each sample rotated the plane of polarized light under specified conditions.

In the original example a one–way analysis of the difference between schizophrenic levels of behaviour was examined. Here an added factor of blocking will be introduced as a 'row' effect where we may assume a blocking by, for example, age of patient.

In the example given by Ramano, although the angle of rotation had been measured a standard arithmetic mean had been calculated and a 'linear' analysis of variance carried out. However, although this is an incorrect approach to the analysis, the estimated concentration parameters are shown to be very large and therefore the Normal approximation to the von Mises distribution may be used. Table 11.3.1 reproduces the data given in Ramano (1977):

Table 11.3.1 Optical Activity Measurements of a Specific Compound Contained in the Brain Fluid of Persons Classified According to Levels of Schizophrenic Behaviour

Blocks	Levels 1	2	3	4	5	Angular Mean
1	11.51°	12.80°	14.98°	15.71°	20.45°	15.09°
2	11.74°	12.49°	12.90°	15.42°	19.42°	14.39°
3	12.07°	12.01°	14.25°	15.77°	20.25°	14.87°
4	13.15°	13.97°	15.27°	15.07°	17.17°	14.92°
Angular Mean	12.12°	12.82°	14.35°	15.49°	19.32°	

Testing the homogeneity of concentration parameters utilises test statistic (4.5.5). A test statistic value of $U_3 = 4.7477$ (distributed as χ^2_4) for the 5 levels and $U_3 = 2.5107$ (distributed as χ^2_3) for the 4 blocks, indicates that the concentration parameters may be regarded as homogeneous for both factors.

The randomised complete block components of variation are given by:

$$k\left[N - \frac{R^2_{..}}{N}\right] = k\left[\sum_{i=1}^{p}\left[\frac{R^2_{i.}}{N_{i.}}\right] - \frac{R^2_{..}}{N}\right] + k\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2_{..}}{N}\right]$$

$$+ k\left[N - \sum_{i=1}^{p}\left[\frac{R^2_{i.}}{N_{i.}}\right] - \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] + \frac{R^2_{..}}{N}\right] \qquad (11.3.1)$$

and the modified test for the null hypothesis of no difference between the q levels will provide an F–ratio:

$$F_{(q-1),(p-1)(q-1)} = \beta\left[\frac{(p-1)(q-1)\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2_{..}}{N}\right]}{(q-1)\left[N - \sum_{i=1}^{p}\left[\frac{R^2_{i.}}{N_{i.}}\right] - \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] + \frac{R^2_{..}}{N}\right]}\right] \quad (11.3.2)$$

For the null hypothesis of no difference between the p blocks

$$F_{(p-1),(p-1)(q-1)} = \beta\left[\frac{(p-1)(q-1)\left[\sum_{i=1}^{p}\left[\frac{R^2_{i.}}{N_{i.}}\right] - \frac{R^2_{..}}{N}\right]}{(p-1)\left[N - \sum_{i=1}^{p}\left[\frac{R^2_{i.}}{N_{i.}}\right] - \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] + \frac{R^2_{..}}{N}\right]}\right] \quad (11.3.3)$$

The analysis of variance for the optical activity measurements is given in Table 11.3.2.

Table 11.3.2 Analysis of Variance Table

Source of Variation	d.f.	Measure of Variation
Due to schizophrenic level	q-1	$\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2_{..}}{N}$
Due to Block	p-1	$\sum_{i=1}^{p}\left[\frac{R^2_{i.}}{N_{i.}}\right] - \frac{R^2_{..}}{N}$
Residual	(q-1)(p-1)	$N - \sum_{i=1}^{p}\left[\frac{R^2_{i.}}{N_{i.}}\right] - \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] + \frac{R^2_{..}}{N}$
Total	N - 1	$N - \frac{R^2_{..}}{N}$

$$\sum_{i=1}^{p}\left[\frac{R_{i.}^2}{N_{i.}}\right] = 19.9957 \qquad \frac{R_{..}^2}{N} = 19.9564$$

$$\sum_{j=1}^{q}\left[\frac{R_{.j}^2}{N_{.j}}\right] = 19.9568 \qquad p = 4 \quad q = 5 \quad N = 20$$

which gives the following ANOVA table.

Table 11.3.3 Analysis of Variance Table

Source of Variation	d.f.	Measure of Variation MV	Mean MV	F
Due to schizophrenic level	4	0.039263	0.009816	30.311
Due to Block	3	0.000414	0.000138	0.426
Residual	12	0.003886	0.0003288	
Total	19	0.043563		

From concentration parameter approximation (3.3.11) $\hat{k} \approx 459$, as $\hat{k}$ is so large the correction factor β may be neglected ($\beta = 1.00043$). Hence $F_{4,12} = 30.311$ and as the table values of $F_{4,12}(0.05) = 3.26$, $F_{4,12}(0.01) = 5.41$, the analysis shows that the optical activity of the compound differs significantly for schizophrenic behaviour recognised by the psychiatrist.

Testing the difference between the blocks gives $F_{3,12} = 0.426$, as the table value of $F_{3,12}(0.05) = 3.49$, the analysis indicates no significant difference between the 4 blocks under investigation.

Example 11.3.2

This example of a randomised complete block analyses a hypothetical data set where the concentration parameter is close to two. As was discussed in Chapter 6, a data set of this nature may not be analysed via extension of the original techniques. Table 11.3.4 shows the hypothetical data set which will be used to illustrate the application and robustness of the new approach.

Table 11.3.4

Block	Treatment 1	2	3	4	5	6	Angular Mean
1	311°	299°	338°	298°	286°	305°	305.966°
2	326°	39°	354°	47°	10°	354°	8.283°
3	10°	45°	309°	319°	25°	339°	345.974°
4	55°	48°	54°	17°	29°	69°	45.440°
Angular Mean	353.98°	7.657°	351.89°	349.795°	2.256°	353.053°	

As with all the previous examples the homogeneity of the concentration parameters must be tested prior to the analysis of variance. Test statistic values of $U_3 = 0.0821$ (distributed as χ^2_5) for the 6 treatments, and $U_3 = 2.7967$ (distributed as χ^2_3) for the 4 blocks, indicates that the concentration parameters may be regarded as homogeneous for both factors.

Using the test statistic (11.3.2) and (11.3.3) and the Analysis of Variance Table 11.3.2 from the previous example the resulting statistics are obtained.

$$\sum_{i=1}^{p} \left[\frac{R_{i.}^2}{N_{i.}}\right] = 20.519145 \qquad \frac{R_{..}^2}{N} = 13.360056$$

$$\sum_{j=1}^{q} \left[\frac{R_{.j}^2}{N_{.j}}\right] = 13.539216 \qquad pq = N = 24$$

which gives the following ANOVA table

Table 11.3.5 Analysis of Variance

Source of Variation	d.f.	Measure of Variation MV	Mean MV	F
Due to treatments	3	0.178658	0.05955	0.2705
Due to Blocks	5	7.158587	1.43172	6.5035
Residual	15	3.302197	0.22015	
Total	23	10.639442		

From concentration parameter approximation (3.2.7) $\hat{k} = 2.34$, hence $\beta = 1.1159$. The modified F' value for the testing of the differing blocks becomes 1.1159 x 6.5035 = 7.257 and as the table values of $F_{5,15}(0.05) = 2.9$ and $F_{5,15}(0.01) = 4.56$, the analysis shows that there is a significant difference between the blocks.

The modified F' value for the testing of the differing treatments becomes 1.1159 x 0.2705 = 0.3019. The table value of $F_{3,15}(0.05) = 3.29$ indicates that there is no significant difference between the observed values within the treatments. Figures 11.3.1 and 11.3.2 illustrate the differences between treatments and blocks, and help to understand and confirm the results obtained. Figure 11.3.1 shows the large spread between the 4 blocks, whilst Figure 11.3.2 shows little spread between the 6 treatments, both confirmed by the analysis of variance

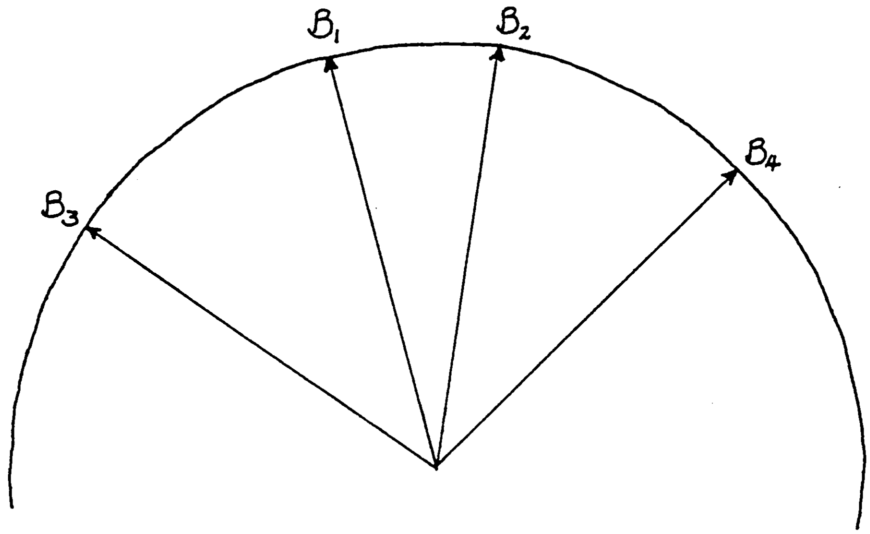


Figure 11.3.1 Angular Mean Responses for the Block Effects

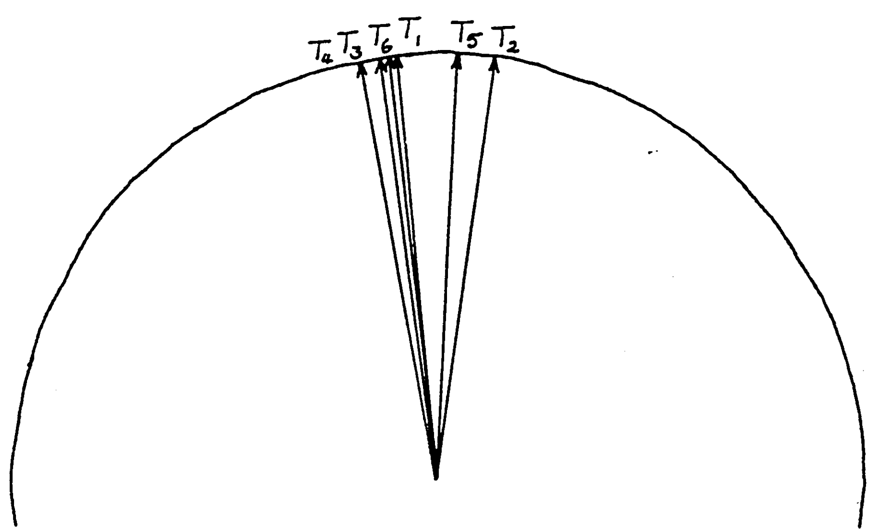


Figure 11.3.2 Angular Mean Responses for the Treatment Effects

Example 11.4.1 for large k

Table 11.4.1, overleaf, gives a hypothetical example data set for the two-way classification design with large k. The data may, for example, be representative of the time of on-set of an illness with relation to new drugs and differing groups of people.

The two-way analysis components of variation are given by:

$$k\left[N - \frac{R^2_{...}}{N}\right] = k\left[\sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i..}}\right] - \frac{R^2_{...}}{N}\right] + k\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j.}}{N_{.j.}}\right] - \frac{R^2_{...}}{N}\right]$$

$$+ k\left[N - \sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R^2_{ij.}}{N_{ij.}}\right]\right]$$

$$+ k\left[\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R^2_{ij.}}{N_{ij.}}\right] - \sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i..}}\right] - \sum_{j=1}^{q}\left[\frac{R^2_{.j.}}{N_{.j.}}\right] + \frac{R^2_{...}}{N}\right] \tag{11.4.1}$$

Testing the homogeneity of concentration parameters between cells produces a test statistic of U_3 = 0.682 (distributed as χ^2_5) indicating the equality of concentration parameters. Table 11.4.2 provides the analysis of variance statistics required.

TABLE 11.4.1 Two-way classification

		FACTOR A			
		A_0	A_1	A_2	
FACTOR B	B_0	283° 304° $R_{11.}=4.892$ 317° $N_{11.}=5$ 291° $\bar{\theta}_{11.}=300.21°$ 306°	289° 255° $R_{12.}=4.891$ 261° $N_{12.}=5$ 277° $\bar{\theta}_{12.}=270.78°$ 272°	2° 322° $R_{13.}=4.809$ 345° $N_{13.}=5$ 352° $\bar{\theta}_{13.}=340.81°$ 323°	$R_{1..}=12.825$ $N_{1..}=15$ $\bar{\theta}_{1..}=303.46°$
	B_1	294° 326° $R_{21.}=4.866$ 295° $N_{21.}=5$ 311° $\bar{\theta}_{21.}=309.61°$ 322°	19° 345° $R_{22.}=4.897$ 359° $N_{22.}=5$ 8° $\bar{\theta}_{22.}=0.98°$ 354°	24° 9° $R_{23.}=4.909$ 41° $N_{23.}=5$ 15° $\bar{\theta}_{23.}=23.17°$ 27°	$R_{2..}=12.825$ $N_{2..}=15$ $\bar{\theta}_{2..}=352.11°$
		$R_{.1.}=9.726$ $N_{.1.}=10$ $\bar{\theta}_{.1.}=304.89°$	$R_{.2.}=6.911$ $N_{.2.}=10$ $\bar{\theta}_{.2.}=315.93°$	$R_{.3.}=9.062$ $N_{.3.}=10$ $\bar{\theta}_{.3.}=2.22°$	$R_{...}=23.197$ $N_{...}=30$ $\bar{\theta}_{...}=327.59°$

Table 11.4.2 Analysis of Variance Table

Source of Variation	d.f.	Measure of Variation
Due to Factor A	q-1	$\sum_{j=1}^{q}\left[\frac{R^2_{.j.}}{N_{.j.}}\right] - \left[\frac{R^2_{...}}{N}\right]$
Due to Factor B	p-1	$\sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i..}}\right] - \left[\frac{R^2_{...}}{N}\right]$
Interaction AB	(p-1)(q-1)	$\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R^2_{ij.}}{N_{ij.}}\right] - \sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i..}}\right] - \sum_{j=1}^{q}\left[\frac{R^2_{.j.}}{N_{.j.}}\right] + \frac{R^2_{...}}{N}$
Residual	pq(l-1)	$N - \sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R^2_{ij.}}{N_{ij.}}\right]$
Total	N - 1	$N - \frac{R^2_{...}}{N}$

$$\sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i..}}\right] = 21.601414 \qquad \frac{R^2_{...}}{N} = 17.935976$$

$$\sum_{j=1}^{q}\left[\frac{R^2_{.j.}}{N_{.j.}}\right] = 22.44784 \qquad p = 2 \quad q = 3 \quad l = 5$$

$$N = pq = 30$$

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R^2_{ij.}}{N_{ij.}}\right] = 28.550744$$

which gives the following ANOVA table:

Table 11.4.3 Analysis of Variance Table

Sources of Variation	d.f.	Measure of Variation MV	Mean MV	F
Due to Factor A	2	4.51186	2.25593	37.35877
Due to Factor B	1	3.66544	3.66544	60.7005
Interaction AB	2	2.43747	1.218735	20.1825
Residual	24	1.44925	0.060385	
Total	29	12.06402		

From concentration parameter approximation (3.3.10) $\hat{k} = 2.57$, hence $\beta = 1.1025$.

The modified test for the null hypothesis of no significant difference between the q levels of Factor A provides the F-ratio:

$$F_{(q-1),pq(l-1)} = \beta \left[\frac{pq(l-1)\left[\sum_{j=1}^{q} \left[\frac{R^2_{.j.}}{N_{.j.}} \right] - \frac{R^2_{...}}{N} \right]}{(q-1)\left[N - \sum_{i=1}^{p} \sum_{j=1}^{q} \left[\frac{R^2_{ij.}}{N_{ij.}} \right] \right]} \right] \qquad (11.4.2)$$

Therefore $F'_{2,24} = 1.1025 \times 37.35877 = 41.188$ and as the table values of $F_{2,24}(0.05) = 3.4$, $F_{2,24}(0.01) = 5.61$, the analysis indicates a very large significant difference between the q levels of Factor A.

Similarly, F–ratios (11.4.3) and (11.4.4) provide the null hypothesis tests with regards to the p levels of Factor B and the pq levels of interaction AB respectively.

$$F_{(p-1),pq(l-1)} = \beta\left[\frac{pq(l-1)\left[\sum_{j=1}^{q}\left[\frac{R_{i..}^2}{N_{i..}}\right] - \frac{R_{...}^2}{N}\right]}{(p-1)\left[N - \sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R_{ij.}^2}{N_{ij.}}\right]\right]}\right] \tag{11.4.3}$$

$$F_{(p-1)(q-1),pq(l-1)} =$$

$$\beta\left[\frac{pq(l-1)\left[\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R_{ij.}^2}{N_{ij.}}\right] - \sum_{i=1}^{p}\left[\frac{R_{i..}^2}{N_{i..}}\right] - \sum_{j=1}^{q}\left[\frac{R_{.j.}^2}{N_{.j.}}\right] + \frac{R_{...}^2}{N}\right]}{(p-1)(q-1)\left[N - \sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R_{ij.}^2}{N_{ij.}}\right]\right]}\right] \tag{11.4.4}$$

$F'_{1,24} = 1.1025 \times 60.7005 = 66.922$ and as the table values of $F_{1,24}(0.05) = 4.26$, $F_{1,24}(0.01) = 7.82$, the analysis shows there to be a very large significant difference between the observed results for Factor B. Testing the difference between the interaction terms AB shows a similarly large significant difference, $F'_{2,24} = 22.25$.

Figures 11.4.1, 11.4.2 and 11.4.3 emphasize the observed differences found between the q levels of Factor A, the p levels of Factor B, and the pq levels of interaction respectively.

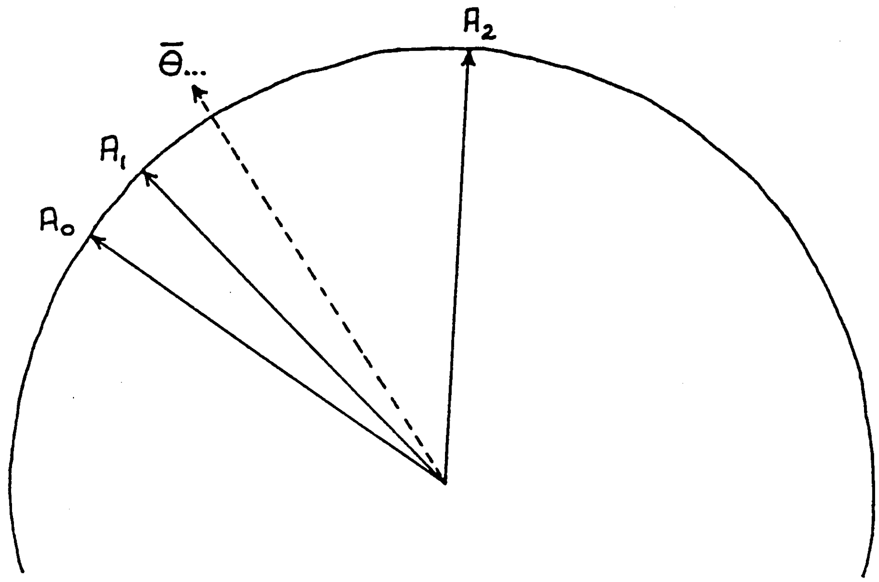


Figure 11.4.1 Angular Differences between the q Levels of Factor A

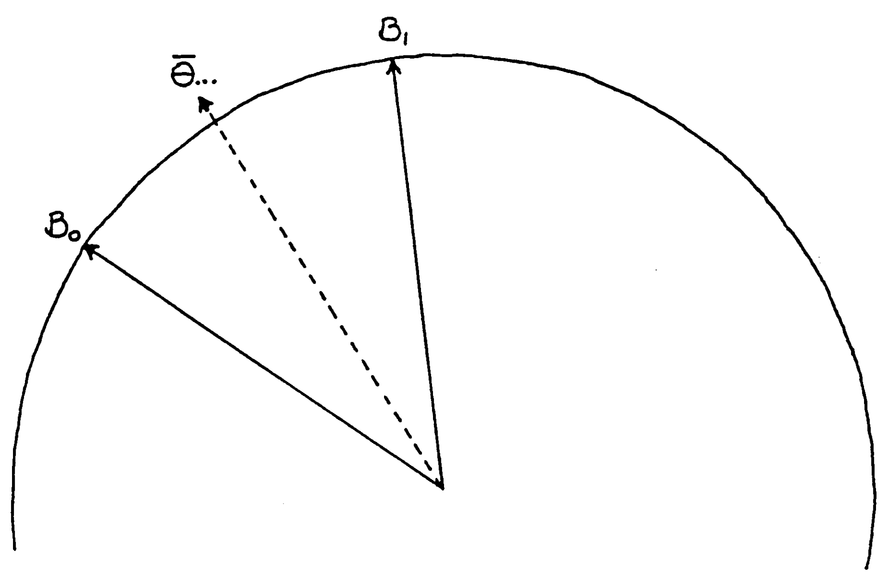
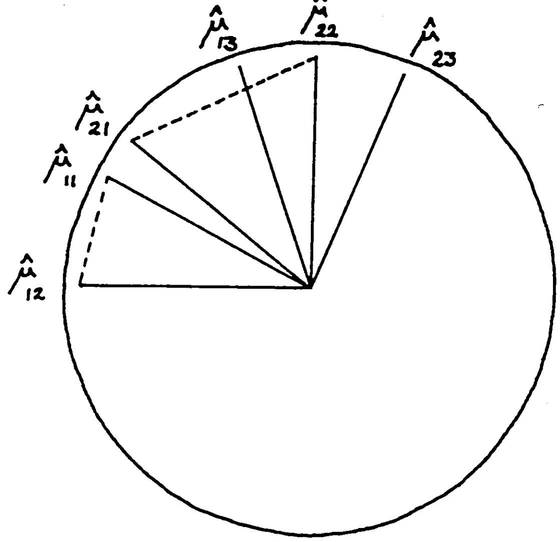
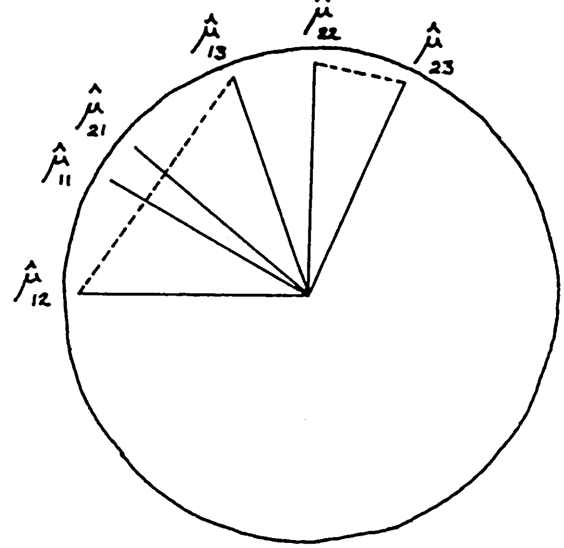


Figure 11.4.2 Angular Differences between the p levels of Factor B

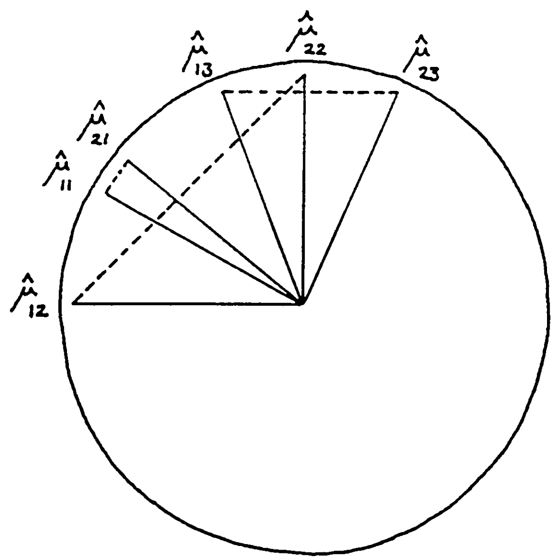
Figures 11.4.1 and 11.4.2 illustrate the angular mean differences between the q and p levels of Factors A and B, the solid lines represent the angular mean for each level, while the dotted line indicates the associated range of values accounting for that mean. The larger significant differences between levels within both factors are clearly seen.



(a) Difference between levels of A_0 and A_1 Factor A



(b) Difference between levels of A_1 and A_2 Factor 2



(c) Difference between levels of B_0 and B_1 Factor B

Figures 11.4.3 Mean Responses Indicating Interaction

Figures (a)–(c) indicate the presence of interaction between the two Factors, represented by the length of the line segments between the differing levels.

The hypothetical example above shows a design data set where all the differing groupings of, cells, rows, columns and overall, produce a large value of concentration parameter. The problems arise, in the majority of cases, where the concentration parameters for the differing groupings do not remain large and equal. As the complexity of designs grow the chance that all the cells, rows, columns etc have the same large and equal concentration parameter becomes highly unlikely. Confusion arises in the original 'simple' one-way analysis when the q samples all have large equal k, but their combined sample gives a small k. Batschelet (1981) assumes that the parameter of concentration has the same value in each population and that $\hat{k}$ is found from the average of the sample resultant lengths, and therefore ignores the combined overall sample. The test statistic derived by Watson and William (1956) is based, however, on the combined overall sample concentration parameter k, and it is this value that should be used. This problem limits the usage of the original test statistic and was highlighted in the data set of Example 7.2.1 where a two-way design was examined. The new generalised approach was constructed in such a manner as not to breakdown under such conditions. The following example re-analyses the data set of Example 7.2.1, this time using the new approach.

Example 11.4.2 for small k

In Example 7.2.1 the possible breakdown of the original extended techniques was discussed, here this data set is re-analysed via the new approach. Using the component statistics of (11.4.1) and the analysis of variance Table 11.4.2 the following ANOVA Table is produced:

Table 11.4.4 Analysis of Variance Table

Source of Variation	d.f.	Measure of Variation
Due to Factor A	1	9.33114
Due to Factor B	1	8.83213
Interaction AB	1	0.02437
Residual	16	1.7936
Total	19	19.98129

$$\sum_{i=1}^{p}\left[\frac{R_{i.}^{2}}{N_{i.}}\right] = 8.85089 \qquad \frac{R_{..}^{2}}{N} = 0.01871$$

$$\sum_{j=1}^{q}\left[\frac{R_{.j}^{2}}{N_{.j}}\right] = 9.34985 \qquad p = 2 \quad q = 2 \quad l = 5$$

$$N = pql = 20$$

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R_{ij}^{2}}{N_{ij}}\right] = 18.2064$$

In Example 7.2.1 the sum of the two mean effects measure of variation was greater than the total measure of variation and consequently the interaction term was negative. The first, and most important, property shown in Table 11.4.4 is that the component measures of variation for the new approach remain positive and sum to the associated total measure of variation.

The concentration parameters of Table 7.2.1 may be shown to be equal within cells and between factors, however, the overall k is very small. Chapter 10 has shown the residual and total measures are not chi-squared distributed when k is small and therefore the F-ratio cannot be computed, and examination of the main effects and interaction must be carried out using the associated chi-squared test statistic for small k.

Test Statistic

$$\hat{k} = 0.0612 \qquad \frac{2}{1-\rho^2} = 2.00187 \quad \text{where } \rho = A(\hat{k}) = \frac{I_1(\hat{k})}{I_0(\hat{k})}$$

Between measure of variation for Factor A

$$\frac{2}{1-\rho^2}\left\{\sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] - \frac{R^2_{..}}{N}\right\} = 18.68$$

This is distributed as $\chi^2_2(q-1)$, from tables $\chi^2_2(0.05) = 5.991$ $\chi^2_2(0.01) = 4.605$, indicating a significant difference between the observed responses of Factor A.

Between measure of variation of Factor B

$$\frac{2}{1-\rho^2}\left\{\sum_{i=1}^{p}\left[\frac{R^2_{i.}}{N_{i.}}\right] - \frac{R^2_{..}}{N}\right\} = 17.68$$

Distributed as $\chi^2_2(p-1)$, once again indicating a significant difference between the observed responses of Factor B.

Between measure of variation for interaction term AB

$$\frac{2}{1-\rho^2}\left\{\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R^2_{ij}}{N_{ij}}\right] - \sum_{i=1}^{p}\left[\frac{R^2_{i.}}{N_{i.}}\right] - \sum_{j=1}^{q}\left[\frac{R^2_{.j}}{N_{.j}}\right] + \frac{R^2_{..}}{N}\right\} = 0.049$$

Distributed as $\chi^2_2(p-1)(q-1)$, indicating that no interaction exists between Factors A and B.

This hypothetical data set was designed solely to emphasize the possible breakdown of the original extended techniques. The main effects were purposely set with maximum distance from each other to produce a large significant difference within both factors.

11.5 The Latin or Graeco-Latin Square

Example 11.5.1

The object of the experiment was to ascertain whether a modified annealing procedure (to heat and then cool slowly to prevent brittleness) could be introduced into the production of light gauge domestic copper tube. The original experiment measured the subsequent tensile strength of the tube (Davies (1963)), here the results will be taken as the breaking angle of the tube. In deciding the form of the tests it is necessary to consider possible causes of variation in the results, including variation in the material itself and variations in temperature over the annealing furnace. Material variations are studied by taking samples of eight tubes at random on each of eight days spread over a period of three weeks, thus allowing normal process variations to be covered adequately. The 64 tubes were held in the furnace in a jig having eight horizontal rows and eight vertical columns of holes i.e. on 8 x 8 square. The construction of the furnace indicated that temperature variations, if present, would be horizontal or vertical, and no appreciable interaction was expected between rows and columns.

The absence of appreciable interaction enabled factors to be examined by means of a Latin or Graeco-Latin Square. For this example the Graeco-Latin arrangement is employed in the experiment, so that two factors each of eight treatments could be allotted to the 64 cells in such a way that the two factors and row and column effects could be determined independently. The two factors were:

1. Day of Manufacture

2. Number allotted to an individual tube within the sample of eight.

From the point of view of this example the second is a dummy factor, since the numbering of the tubes did not correspond to any physical reality.

A separate square selected at random was used for each temperature, and the rows, columns, and letters were themselves randomised. The results for a nominal temperature of 300°C are shown in Table 11.5.1, in which the rows and columns represent positions in the furnace, the letters A to H the day of manufacture, and the numbers 1 to 8 the designation within the sample. The figures in brackets are the associated breaking angles.

The Latin or Graeco-Latin square components are given by:

$$k\left[N - \frac{R^2_{\cdots\cdot}}{N}\right] = \underbrace{k\left[\sum_{i=1}^{p}\left[\frac{R^2_{i\cdots}}{N_{i\cdots}}\right] - \frac{R^2_{\cdots\cdot}}{N}\right]}_{\text{Measure of Variation due to row of jig}} + \underbrace{k\left[\sum_{i=1}^{p}\left[\frac{R^2_{\cdot j\cdot\cdot}}{N_{\cdot j\cdot\cdot}}\right] - \frac{R^2_{\cdots\cdot}}{N}\right]}_{\text{Measure of Variation due to column of jig}}$$

Total measure of variation

$$+ \underbrace{k\left[\sum_{l=1}^{p}\left[\frac{R^2_{\cdot\cdot l\cdot}}{N_{\cdot\cdot l\cdot}}\right] - \frac{R^2_{\cdots\cdot}}{N}\right]}_{\text{Measure of Variation due to day of manufacture}} + \underbrace{k\left[\sum_{a=1}^{p}\left[\frac{R^2_{\cdots a}}{N_{\cdots a}}\right] - \frac{R^2_{\cdots\cdot}}{N}\right]}_{\text{Measure of Variation due to number designation within the sample}}$$

$$+ \underbrace{k\left[N - \sum_{i=1}^{p}\left[\frac{R^2_{i\cdots}}{N_{i\cdots}}\right] - \sum_{j=1}^{p}\left[\frac{R^2_{\cdot j\cdot\cdot}}{N_{\cdot j\cdot\cdot}}\right] - \sum_{l=1}^{p}\left[\frac{R^2_{\cdot\cdot l\cdot}}{N_{\cdot\cdot l\cdot}}\right] - \sum_{a=1}^{p}\left[\frac{R^2_{\cdots a}}{N_{\cdots a}}\right] + 3\frac{R^2_{\cdots\cdot}}{N}\right]}_{\text{Residual Measure of Variation}} \qquad (11.5.1)$$

Carrying out the analysis of variance we obtain Table 11.5.2.

TABLE 11.5.1

Tests on annealing light gauge domestic copper tubes

Column	1	2	3	4	5	6	7	8	Mean Deviation
Row									
1	D3(16.6)	H4(16.9)	C5(17.4)	B6(17.4)	E8(15.8)	A1(18.2)	G2(15.7)	F7(15.8)	16.725
2	F6(15.9)	E5(16.4)	G4(15.8)	A3(19.0)	H2(17.6)	B7(17.8)	C8(18.9)	D1(17.1)	17.312
3	B5(17.1)	C6(16.8)	H3(19.2)	D4(16.6)	G1(15.8)	F8(17.8)	E7(18.4)	A2(18.3)	17.500
4	A4(17.7)	G3(15.9)	E6(16.3)	F5(16.0)	C7(17.6)	D2(17.8)	H1(18.1)	B8(16.5)	16.987
5	C1(17.4)	B2(17.0)	D8(16.8)	H7(19.2)	A5(20.3)	E3(18.4)	F4(15.9)	G6(15.7)	17.587
6	E2(16.5)	F1(16.0)	A7(16.9)	G8(15.9)	D6(17.1)	C4(17.5)	B3(17.4)	H5(19.6)	17.112
7	G7(15.8)	A8(16.9)	F2(15.9)	E1(16.5)	B4(17.6)	H6(19.4)	D5(17.1)	C3(18.3)	17.187
8	H8(18.6)	D7(17.4)	B1(17.4)	C2(19.2)	F3(16.8)	G5(15.7)	A6(17.4)	E4(18.4)	17.612
Mean Deviation	16.950	16.662	16.962	17.475	17.325	17.825	17.362	17.462	17.253

Lot	A	B	C	D	E	F	G	H	Overall
Mean Deviation	18.087	17.275	17.887	17.062	17.087	16.262	15.787	18.575	17.253
Number	1	2	3	4	5	6	7	8	
Mean Deviation	17.062	17.250	17.700	17.050	17.450	17.000	17.362	17.150	

Table 11.5.2 Analysis of Variance of Data from Table 11.5.1

Source of Variation	d.f.	Measure of Variation MV	Mean MV	F
Between rows of jig	7	1.657×10^{-3}	2.367×10^{-4}	1.5
Between columns of jig	7	2.347×10^{-3}	3.353×10^{-4}	2.12*
Between positions	14	4.004×10^{-3}	2.86×10^{-4}	1.81*
Between lots (letters)	7	1.4717×10^{-2}	2.102×10^{-3}	13.32**
Between numbers	7	9.83×10^{-4}	1.404×10^{-4}	
	= 42	$= 6.629 \times 10^{-3}$	$= 1.578 \times 10^{-4}$	
Residuals	35	5.646×10^{-3}	1.613×10^{-4}	
Total	63	2.535×10^{-2}		

$$\sum_{i=1}^{p} \left[\frac{R^2_{i\ldots}}{N_{i\ldots}}\right] = 63.976307 \qquad \frac{R^2_{\ldots\ldots}}{N} = 63.97465$$

$$\sum_{j=1}^{p} \left[\frac{R^2_{.j..}}{N_{.j..}}\right] = 63.976997 \qquad N = 64$$

$$\sum_{l=1}^{p} \left[\frac{R^2_{..l.}}{N_{..l.}}\right] = 63.989367 \qquad p = 8$$

$$\sum_{a=1}^{p} \left[\frac{R^2_{\ldots a}}{N_{\ldots a}}\right] = 63.975633$$

The concentration parameter approximation is so large the correction factor β may be neglected (β = 1)

$F_{7,42}(0.10) = 1.87$ $F_{14,42}(0.10) = 1.66$

$F_{7,42}(0.05) = 2.25$ $F_{14,42}(0.05) = 1.92$

$F_{7,42}(0.01) = 3.12$ $F_{14,42}(0.01) = 2.52$

* = significant at the 10% level ** = highly significant (1% level)

As expected, the mean measure of variation of the factor representing number designation within the sample does not differ significantly from the mean measure of variation. The number factor may therefore be combined with that for the residual to provide a new estimate of error with 42 degrees of freedom.

Variation between columns (i.e. across the furnace) and total variation among positions (variation over all positions in the furnace) may be judged significantly different at the 10 percent level. Clearly the major factor causing variation in the breaking angle was variation in the lots of tubes; position in the furnace is of little or no importance.

Although the theory behind a design such as the split plot has not been derived and fully checked, the new generalised approach is robust enough (particularly for large k) to analyse any experimental design if the components are adapted appropriately and the assumptions are checked sufficiently.

Example 11.6.1

In an experiment on the preparation of chocolate cakes three recipes for preparing the mixture were compared. Recipes I and II differed in that the chocolate was added at 40°C and 60°C, respectively, while Recipe III contained extra sugar. For each recipe enough mixture was made for six cakes. These 18 cakes were then placed in an oven which was then heated slowly. When the temperature had reached 175°C three cakes, one from each recipe, were selected at random for removal, another three at 185°C, and so on until the last three cakes were removed at 225°C. In this manner the recipes are representative of the "whole-unit" treatments, while the baking temperatures are representative of the "sub-unit" treatments. There were 7 replications, one replication was completed before starting the next, so that differences among replicates represented a time difference.

A number of measurements were made on the cakes. Table 11.6.1 presents the measurements of the breaking angle. One half of the cake was held fixed, while the other half was pivoted about the middle until breakage occured.

The split components of variation are given by:

$$k\left[N - \frac{R^2_{...}}{N}\right] = k\left[\sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i..}}\right] - \frac{R^2_{...}}{N}\right] + k\left[\sum_{j=1}^{q}\left[\frac{R^2_{.j.}}{N_{.j.}}\right] - \frac{R^2_{...}}{N}\right]$$

$$+ k\left[\sum_{l=1}^{m}\left[\frac{R^2_{..l}}{N_{..l}}\right] - \frac{R^2_{...}}{N}\right]$$

$$+ k\left[\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R^2_{ij.}}{N_{ij.}}\right] - \sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i..}}\right] - \sum_{j=1}^{q}\left[\frac{R^2_{.j.}}{N_{.j.}}\right] + \frac{R^2_{...}}{N}\right]$$

$$+ k\left[\sum_{i=1}^{p}\sum_{l=1}^{m}\left[\frac{R^2_{i.l}}{N_{i.l}}\right] - \sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i..}}\right] - \sum_{l=1}^{m}\left[\frac{R^2_{..l}}{N_{..l}}\right] + \frac{R^2_{...}}{N}\right]$$

$$+ k\left[\sum_{j=1}^{q}\sum_{l=1}^{m}\left[\frac{R^2_{.jl}}{N_{.jl}}\right] - \sum_{j=1}^{q}\left[\frac{R^2_{.j.}}{N_{.j.}}\right] - \sum_{l=1}^{m}\left[\frac{R^2_{..l}}{N_{..l}}\right] + \frac{R^2_{...}}{N}\right]$$

$$+ k\left[N + \sum_{i=1}^{p}\left[\frac{R^2_{i..}}{N_{i..}}\right] + \sum_{j=1}^{q}\left[\frac{R^2_{.j.}}{N_{.j.}}\right] + \sum_{l=1}^{m}\left[\frac{R^2_{..l}}{N_{..l}}\right] - \sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{R^2_{ij.}}{N_{ij.}}\right]\right.$$

$$\left. - \sum_{i=1}^{p}\sum_{l=1}^{m}\left[\frac{R^2_{i.l}}{N_{i.l}}\right] - \sum_{j=1}^{q}\sum_{l=1}^{m}\left[\frac{R^2_{.jl}}{N_{.jl}}\right] - \frac{R^2_{...}}{N}\right] \qquad (11.6.1)$$

Carrying out the analysis of variance we obtain Table 11.6.2.

Table 11.6.1 Examination of Breaking Angles

	Replications	Temperature 175°	185°	195°	205°	215°	225°	Angular Mean
Recipe I	1	42	46	47	39	53	42	35.37°
	2	47	29	35	47	57	45	
	3	32	32	37	43	45	45	
	4	26	32	35	24	39	26	
	5	28	30	31	37	41	47	
	6	24	22	22	29	35	26	
	7	26	23	25	27	33	35	
Recipe II	1	39	46	51	49	55	42	36.02°
	2	35	46	47	39	52	61	
	3	34	30	42	35	42	35	
	4	25	26	28	46	37	37	
	5	31	30	29	35	40	36	
	6	24	29	29	29	24	35	
	7	22	25	26	26	29	36	
Recipe III	1	46	44	45	46	48	63	36.14°
	2	43	43	43	46	47	58	
	3	33	24	40	37	41	38	
	4	38	41	38	30	36	35	
	5	21	25	31	35	33	23	
	6	24	33	30	30	37	35	
	7	20	21	31	24	30	33	
Angular Mean		31.42°	32.22°	35.33°	35.85°	40.66°	39.62°	35.84°

Table 11.6.1 Continued

Replication	1	2	3	4	5	6	7
Angular Mean	46.82°	45.55°	36.95°	33.28°	32.38°	28.72°	27.33°

Table 11.6.2 Analysis of Variance of Data from Table 11.6.1

Source of Variation	d.f.	Measure of Variation MV	Mean MV	F
Between Recipes	2	0.00207	0.001035	0.081
Between Replications	6	1.92835	0.3213916	25.169*
Interaction Recipe/ Replication	12	0.15323	0.0127691	
Between Temperatures	5	0.44041	0.088082	13.47*
Interaction Recipes/ Temperatures	30	0.17493	0.005831	0.677
Residual	60	0.51635	0.00860583	
Total	125	3.28073		

$$\sum_{i=1}^{p} \left[\frac{R^2_{i..}}{N_{i..}}\right] = 122.72134 \qquad \sum_{i=1}^{p} \sum_{j=1}^{q} \left[\frac{R^2_{ij.}}{N_{ij.}}\right] = 123.22714$$

$$\sum_{j=1}^{q} \left[\frac{R^2_{.j.}}{N_{.j.}}\right] = 123.15968 \qquad \sum_{i=1}^{p} \sum_{l=1}^{m} \left[\frac{R^2_{i.l}}{N_{i.l}}\right] = 124.80292$$

$$\sum_{l=1}^{m} \left[\frac{R^2_{..l}}{N_{..l}}\right] = 124.64762 \qquad \sum_{j=1}^{q} \sum_{l=1}^{m} \left[\frac{R^2_{.jl}}{N_{.jl}}\right] = 125.26296$$

$$\frac{R^2_{...}}{N} = 122.71927 \qquad p = 3 \qquad q = 6 \qquad m = 7$$

$$N = pqm = 126$$

* = Highly significant (1% level)

The concentration parameter approximation is large and therefore the correction factor β may be neglected (β = 1)

From tables $F_{2,12}(0.05) = 3.89$

$F_{6,12}(0.05) = 3.00$ $\qquad F_{6,12}(0.01) = 4.82$

$F_{5,10}(0.05) = 3.33$ $\qquad F_{5,10}(0.01) = 5.64$

$F_{30,60}(0.05) = 1.65$

The analysis of variance Table 11.6.2, indicates that there was no significant difference observed between the recipes. This is shown more clearly from examination of the angular mean breaking angles associated with the three recipes, given in Table 11.6.1, where little difference may be noted. Variation between replicates and temperatures, however, are significantly large and are the dominating factors causing variation in the breaking angle. The angular mean breaking angle is seen to increase as the temperature of the oven increases and may be an understandable effect. The angular mean breaking angle of the replications, however, show an even clearer decrease in value as the experiments are carried out. This heavy dependence on time is less understandable and may indicate, for example, that another factor or condition may be affecting the oven state.

11.7 Summary

This chapter has shown how the new generalised approach may be applied to real situations where directional data values are measured. The effect of large and small concentration parameters have been emphasized together with the requirement to test for the homogeneity of concentration parameters. The Graeco-latin square and split plot examples have helped to show the suitability of the approach for many experimental design situations and indicate the method by which further test statistics may be built.

CHAPTER 12

SYNOPSIS OF RESULTS AND CONCLUSIONS

The aims of this study have been to extend the knowledge of the present methods of analysis of directional data and to develop suitable analysis of variance techniques for differing experimental design models. These objectives have been achieved, although it has not been possible to produce a single unified approach for both large and small concentration parameter. However, separate techniques were produced under a generalised approach which has been shown to be applicable to many experimental design problems. To obtain this approach the work has followed several steps individually discussed within the thesis.

Before discussing the development of the new techniques an understanding of the development, constructions and distributional form of the original methods was required. Within the first four chapters a simple yet informative review of the many approximations for the concentration parameter k, both large and small, was given. This work included the plotting of the residuals and relative residuals for each approximation to indicate the accuracy and range of application for the statistics. Several were shown to be inappropriate despite their complex form. The analysis culminated in a summary table of the 'best' and 'best simple' approximations for both large and small k (Table 3.5.1) and indicated the required approximations for use with the techniques to be developed.

Chapter 5 discussed a generalised linear modelling approach, analogous to the normal theory of linear regression, in order to estimate the individual parameter values for application to the maximum likelihood method. The work showed how the observations may be 'added together' under the constraint that the sum of the sines of the factor effects equals zero. Using this fact parameter estimates may be found for the one-way analysis of variance giving further understanding to the underlying structure of the data. The approach, under the above constraint, was found to produce the original one-way analysis of variance technique developed by Watson and Williams. When applied to larger experimental design problems, however, the optimisation of the circular constrained simultaneous equations could not be found, without very good initial estimates, due to numerous local maxima. Further work in this particular area, using improved techniques for convergence and ever increasing improvements in computing methods, is worthy of investigation.

Although it would have been useful to have produced a computer optimisation program to solve the constrained equations, the constant need for a computer program as a general method for analysing circular experimental designs, would be rather restrictive. Therefore, a simple construction of an analysis of variance was still required. Chapter 6 investigated the possibility of extending the original approach, with large k, for other designs such as the nested or hierarchical, the randomised complete block and the two-way design with interaction. The methods were seen to extend for these designs ($k > 2$) and with good chi-squared approximations. However the assumption of equal and large k must hold, not only for the cell, column and row observations, but for the overall sample. This, under larger and larger designs i.e. with an increasing number of factors, is extremely restrictive. For example, in situations where the individual row and column factors have large concentration parameters and the mean directions differ, the overall concentration parameter may be small. Under these conditions within a two-way design the 'sum of squares' for the factor effects may add to a value greater than the total 'sum of squares'. A full investigation followed in Chapter 7 where the one-way analysis was reconstructed via a regression approach using basic vector analysis to examine the make-up of the individual components. This brought to light the possible lack of independence between the model components and the non-zero existence of a cross-product term. This questioned not only the use of the original techniques for larger designs but the adequacy of the original one-way analysis.

The development of the new approach needed to overcome the faults of the original techniques but be relatively simple and, if possible, be capable of generalising across all designs. The vector approach used to investigate the original techniques was used again to construct the new test statistics. The method utilizes the resultant lengths associated with each sample mean direction, in order that when sample means are combined the overall mean direction is still obtained. The vector approach minimised the chord distance between mean directions to construct the test statistics. Nevertheless, this is not only testing the difference between mean directions but the associated concentration parameters as well. Hence, a separate examination of the individual concentration parameters <u>must</u> also be carried out for a valid test of the mean directions to be feasible.

The beauty of the construction of components in this manner is its simplicity and, in comparison to the original techniques the independence of the individual components and the zero cross product terms produced. The interpretation and calculation of interaction is discussed and the generalised nature of the technique is illustrated in the construction and proof of the interaction component.

As with Stephens (1969) and Upton (1970) an attempt was made to evaluate numerically the exact distribution of the old and new test statistics. However, even for the simplest single sample test statistics the numerical integration involved was found to be very tedious.

R does not have a simply stated density function, and indeed a direct computation of the significance points of R/N or R^2/N is not simple.

The majority of the tests are complicated by involving R_1, R_2 and R, and, since these statistics are not for the most part independent of one another, either one or another of the conditional distributions needs to be employed. These distributions are exceedingly complicated as are the relevant bounds for integration, and no results of any use were obtained by attempting to integrate numerically.

Alternative investigations were carried out using the power series expansion of ρ to examine the moments of the test statistics to compare with their associated chi-squared distributions. This showed good approximations to the first moment but worsening accuracy to higher moments, depending on the size of k. In a similar manner to the adjustment advocated by Stephens, an important factor is produced to increase the test statistics accuracy.

The simulation techniques, discussed in Appendix B, have been successful in obtaining the characteristics of the von Mises distribution and hence the various test statistics. Elaborate examination of the components for varying designs, concentration parameter and sample size have been carried out together with tests of the statistics power and robustness in comparison to any available alternative techniques. Compared to the alternatives in the one-way analysis the new technique is seen as slightly less accurate and powerful, although it does possess desirable properties as previously described.

For small concentration parameter the new approach, as with alternative tests, can only examine the between measure of variation, seen to be chi-squared distributed. No account can therefore be taken of the residual variation.

Following further justification for the new approach via examination of the randomised complete block and two-way analysis designs, the test statistics were successfully applied to differing problems with varying concentration parameter.

APPENDIX A

INDEX OF NOTATION

$A(k)$ or ρ	:	ratio of $I_1(k)$ to $I_0(k)$
C	:	sum of cosines of angles
$\bar{C}$	:	mean of cosines of angles
$F(\theta)$	:	distribution function
$f(\theta)$	:	circular density
H_0, H_1	:	null and alternative hypothesis
$I_p(k)$	:	modified Bessel function of the first kind and order p
i	:	subscript ranging from 1,2,.....,p
j	:	subscript ranging from 1,2,.....,q
l	:	subscript ranging from 1,2,.....,m
m	:	number of levels for factor 3
N	:	size of sample
p	:	number of levels for factor 1
q	:	number of levels for factor 2
R	:	the resultant length
R_i	:	R for the ith sample
r	:	mean resultant length of sample
S	:	sum of sines of angles
$\bar{S}$	:	mean of sines of angles
$VM(\mu_0,k)$	:	von Mises distribution with mean direction μ_0 and concentration parameter k
β	:	improvement factor for the new approach
γ	:	Stephens improvement factor
k	:	parameter of concentration

μ_0 : population mean direction

σ : standard deviation

θ : circular random variable

$\overline{\theta}$ or $\overline{\theta}_{..}$: overall mean direction of sample

General Tabular Notation

	FACTOR A			
Factor B	A_0	A_1	A_2	
B_0	θ_{111} θ_{112} R_{11} θ_{113} N_{11} θ_{114}	θ_{121} θ_{122} R_{12} θ_{123} N_{12} θ_{124}	θ_{131} θ_{132} R_{13} θ_{133} N_{13} θ_{134}	$R_{1.}$ $N_{1.}$
B_1	θ_{211} θ_{212} R_{21} θ_{213} N_{21} θ_{214}	θ_{221} θ_{222} R_{22} θ_{223} N_{22} θ_{224}	θ_{231} θ_{232} R_{23} θ_{233} N_{23} θ_{234}	$R_{2.}$ $N_{2.}$
	$R_{.1}$ $N_{.1}$	$R_{.2}$ $N_{.2}$	$R_{.3}$ $N_{.3}$	$R_{..}$ N

where

R_{ij} = the resultant length of cell observations in row i (i=1,2,.....,p) and the column j (j=1,2,.....,q)

$R_{i.}$ = the resultant length of all observations in row i (i=1,2,.....,p)

$R_{.j}$ = the resultant length of all observations in column j (j=1,2,.....,q)

$R_{..}$ = the resultant length of all observations in the sample

N = the total number of observations in the sample

APPENDIX B

THE SIMULATION AND ACCURACY OF NUMERICAL RESULTS

All the numerical results presented in this thesis are the product of simulations; all simulations carried out had a minimum of 10,000 samples. For standard simulation a random number z is generated in the range 0 to 1, for a distribution function $F(\theta)$ the number z corresponds to an observation θ from this distribution which is the solution of the equation

$$F(\theta) = z$$

For many distributions this equation can be inverted directly to obtain

$$\theta = F^{-1}(z)$$

but this is not possible for the von Mises distribution. In addition generation of a pseudo-random observation from $VM(0,k)$ cannot be obtained by a simple transformation of $VM(0,1)$ to $VM(0,k)$ and an alternative procedure was used.

The approach used was to fit a probability density function as an envelope around the von Mises distribution to give an acceptance – rejection method which is both simple to program and fast for all values of the concentration parameter. Initially the simplest p.d.f. used was the Uniform function, simple but very slow. Best and Fisher (1979) produced an algorithm to simulate samples from the von Mises distribution using the wrapped Cauchy density (Equation 1.3.5) as the p.d.f. for the envelope. Let $f(x)$ be the p.d.f. of a random variable x which is to be sampled. Let Y be a random variable with p.d.f. proportional to $g(x)$, an upper envelope for $f(x)$ (i.e. $g(x) \geq f(x)$) and let U be a $U(0,1)$ random variable. If (y,u) is a realization of (Y,U), y is accepted as a realization of x if $f(y)/g(y) > u$. The distribution of the accepted values is then exactly the required distribution. Once the observations from the distribution specified by the null hypothesis are generated 10,000 sets of samples of various size and various concentration parameters may be grouped and analysed as required. The homogeneity of concentration parameters was tested prior to all analyses.

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